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Low-amplitude instability as a premise for the spontaneous symmetry breaking in the new integrable semidiscrete nonlinear system

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ABSTRACT

The new integrable semidiscrete multicomponent nonlinear system characterized by two coupling parameters is presented. Relying upon the lowest local conservation laws the concise form of the system is given and its selfconsistent symmetric parametrization in terms of four independent field variables is found. The comprehensive analysis of quartic dispersion equation for the system low-amplitude excitations is made. The criteria distinguishing the domains of stability and instability of low-amplitude excitations are formulated and a collection of qualitatively distinct realizations of a dispersion law are graphically presented. The loop-like structure of a low-amplitude dispersion law of reduced system emerging within certain windows of adjustable coupling parameter turns out to resemble the loop-like structure of a dispersion law typical of beam-plasma oscillations. Basing on the peculiarities of low-amplitude dispersion law as the function of adjustable coupling parameter it is possible to predict the windows of spontaneous symmetry breaking even without an explicit knowledge of the system Lagrangian function. Having been rewritten in terms of properly chosen modified field variables the reduced four wave integrable system can be qualified as consisting of two coupled nonlinear lattice subsystems, namely the self-dual ladder network and the vibrational ones.

1. Introduction

Since the discovery of first integrable nonlinear dynamical models on a regular one-dimensional lattice [1–4] the interest to the development of new integrable semidiscrete nonlinear systems has been steadily supported by the wide range of physical problems, where the spatial discreteness and regularity play a crucial role. Among the most typical physical objects, where the semidiscrete nonlinear systems found their applications, are the optical waveguide arrays [5], semiconductor superlattices [6,7], electric superstructures [8] as well as the regular macromolecular structures of both natural [9] and synthetic [10] origin.

Evidently the more complex nonlinear physical phenomenon requires the more rich nonlinear model for its adequate description. The richness of semidiscrete integrable nonlinear system is dictated by the order of auxiliary spectral matrix $L(n|z)$ consistent with some evolution matrix $A(n|z)$ in the framework of system zero-curvature representation

$$\dot{L}(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z). \tag{1.1}$$

Here the dot written over the matrix $L(n|z)$ in the left-hand side of zero-curvature equation (1.1) means the differentiation with respect to time τ, the integer n denotes the discrete spatial coordinate running from minus to plus infinity, while z denotes the auxiliary spectral parameter independent of time: $\dot{z} = 0$.

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According to Caudrey [11,12] the order of spectral matrix $L(n|z)$ is determined by the number of *distinct* eigenvalues of either of limiting spectral matrices $L^{-}(z) = \lim_{n\to-\infty} L(n|z)$ or $L^{+}(z) = \lim_{n\to+\infty} L(n|z)$. The order of $L(n|z)$ depends both on the rank of limiting spectral matrix and on its matrix structure. Here it is worth noticing that the auxiliary spectral problems linked with known multicomponent semidiscrete nonlinear Schrödinger systems [13–16] taken at vanishing boundary conditions must be treated as the second-order ones despite being rather sophisticated matrix generalizations of the basic Ablowitz–Ladik spectral problem [3,4]. The similar property is typical of the auxiliary spectral problems associated with the matrix generalizations [17,18] of nonlinear Toda system [1,2]. From the physical standpoint the generalized systems in both of just mentioned examples do not acquire the brand-new physical quality inasmuch as the number of parameters responsible for the nonlinear couplings remains the same as in their prototype twins. In the case of semidiscrete nonlinear Schrödinger systems this statement can be confirmed by the direct consideration of low-amplitude normal modes exhibiting essentially the same dependences on wave vector both in the prototype and generalized systems despite the effect of parallel splitting admissible in the latter ones.

Meanwhile recently we have suggested early unknown semidiscrete integrable nonlinear systems [19,20] associated with the fourth order spectral problem, whose spectral matrix reads as follows

$$L(n|z) = \begin{pmatrix} 0 & t_{12}(n) & u_{13}(n)z^{-1} & 0 \\ t_{21}(n) & r_{22}(n)z^2 + t_{22}(n) & s_{23}(n)z + u_{23}(n)z^{-1} & s_{24}(n)z \\ u_{31}(n)z^{-1} & s_{32}(n)z + u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} & t_{34}(n) \\ 0 & s_{42}(n)z & t_{43}(n) & 0 \end{pmatrix}. \tag{1.2}$$

Each of the systems is characterized by several coupling parameters and appears to have all chances to manifest the effect of spontaneous symmetry breaking (in a sense adopted in the theory of fields [21,22]) playing the fundamental role in many branches of physics. In view of the very complicated structure of above systems we have decided to verify the idea about symmetry breaking on a more simple but new and still integrable nonlinear system characterized at least by two coupling parameters.

The first step in this direction was to obtain an appropriate new semidiscrete integrable nonlinear system in the framework of zero-curvature scheme seeking the auxiliary spectral matrix as the third-order one. In so doing it was reasonable to keep some elements of succession between the antecendent [19,20] and sought-for schemes otherwise the procedure of empirical selection of auxiliary spectral matrix consistent with a proper auxiliary evolution matrix in the framework of zero-curvature approach may fail to be fruitful (see expressions (1.2) and (2.2) for the previous $L(n|z)$ and new $M(n|z)$ spectral operators for comparison). The above observation, when combined with the Caudrey definition of the order of a spectral operator [11,12,23], has allowed us to reveal the constructive version of early unknown third-order auxiliary spectral matrix (2.2) giving rise to new integrable systems.

Having been restricted to the reduced semidiscrete nonlinear integrable system in symmetric parametrization we have carried out the comprehensive analysis of its low-amplitude excitations under assumption of real-valued adjustable coupling parameter. Namely, the linear analysis constitutes the second step of our investigation allowing to detect the windows of spontaneous symmetry breaking in each of two reduced nonlinear system under study. The approach does not operate with the system Lagrangian function whose sole isolation seems to be an essentially nontrivial task. The same linear analysis is expected to be helpful in detecting all qualitatively distinct regimes of nonlinear (soliton) dynamics predetermined by the distinct intervals of adjustable coupling parameter framed by the critical points.

2. Zero-curvature equation and mutually consistent auxiliary matrices

In order to ensure the integrability of desired nonlinear system one need to approbate the zero-curvature equation [24]

$$\dot{M}(n|z) = B(n+1|z)M(n|z) - M(n|z)B(n|z) \tag{2.1}$$

by the spectral $M(n|z)$ and evolution $B(n|z)$ matrices chosen properly among the square matrices assumed as Laurent polynomials of spectral parameter z.

The arguments given in Introduction prompt us to define the spectral matrix $M(n|z)$ as the following 3×3 matrix

$$M(n|z) = \begin{pmatrix} z^2 + T(n) & \beta F_+(n)z + \alpha F_+(n) & G_+(n)z + G_-(n)z^{-1} \\ \alpha F_-(n)z + \beta F_-(n) & 0 & \alpha F_-(n) + \beta F_-(n)z^{-1} \\ G_-(n)z + G_+(n)z^{-1} & \beta F_+(n) + \alpha F_+(n)z^{-1} & T(n) + z^{-2} \end{pmatrix} \tag{2.2}$$

and to seek the evolution matrix $B(n|z)$ in the form

$$B(n|z) = \begin{pmatrix} a(n)z^2 + d(n) & \beta b_+(n)z + \alpha b_+(n) & c_+(n)z + c_-(n)z^{-1} \\ \alpha b_-(n)z + \beta b_-(n) & d(n) - c(n) & \alpha b_-(n) + \beta b_-(n)z^{-1} \\ c_-(n)z + c_+(n)z^{-1} & \beta b_+(n) + \alpha b_+(n)z^{-1} & d(n) + a(n)z^{-2} \end{pmatrix}, \tag{2.3}$$

where α and β are some fitting parameters independent of time. Then the direct calculations based on the zero-curvature equation (2.1) confirm our conjecture and permit to decipher almost all matrix elements $B_{jk}(n|z)$ of the tested evolution matrix $B(n|z)$ through the matrix elements $M_{jk}(n|z)$ of chosen spectral matrix $M(n|z)$ provided

$$\alpha^2 + \beta^2 = 0. \tag{2.4}$$

Thus, for the functions entering into the evolution matrix $B(n|z)$ we have

$$a(n) = k, \tag{2.5}$$
$$b_+(n) = kF_+(n), \tag{2.6}$$
$$b_-(n) = kF_-(n-1), \tag{2.7}$$
$$c_+(n) = kG_+(n), \tag{2.8}$$
$$c_-(n) = kG_-(n-1), \tag{2.9}$$
$$d(n) = -k\alpha\beta F_+(n)F_-(n-1) - kG_+(n)G_-(n-1), \tag{2.10}$$

where the summation quantity k can be thought as an arbitrary function of time τ. The only exception is the sampling function $c(n)$ which similarly to other precedents [25,26] remains arbitrary for the time being.

In what follows we equalize the summation function to unity: $k = 1$, inasmuch as that does not lead to the any loss of generality.

3. Evolution equations for the prototype field variables

We call the quantities $F_+(n), F_-(n), G_+(n), G_-(n)$ and $T(n)$ to be the prototype field variables. According to the zero-curvature equation (2.1) and the expressions (2.2) and (2.3)–(2.10) for the spectral $M(n|z)$ and evolution $B(n|z)$ matrices the evolution of these variables is described by the following collection of semidiscrete nonlinear equations

$$\frac{d}{d\tau}\ln[F_+(n)] = G_+(n+1) - G_+(n) - T(n) - \alpha\beta F_+(n+1)F_-(n) - G_+(n+1)G_-(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1) + c(n), \tag{3.1}$$

$$\frac{d}{d\tau}\ln[F_-(n)] = G_-(n) - G_-(n-1) + T(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1) - \alpha\beta F_+(n+1)F_-(n) - G_+(n+1)G_-(n) - c(n+1), \tag{3.2}$$

$$\frac{d}{d\tau}\ln[1 - T(n) + G_+(n) - G_-(n)] = G_+(n) - G_+(n+1) + G_-(n) - G_-(n-1) - \alpha\beta F_+(n+1)F_-(n) - G_+(n+1)G_-(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1), \tag{3.3}$$

$$\frac{d}{d\tau}\ln[1 + T(n) - G_+(n) - G_-(n)] = G_+(n) - G_+(n+1) - G_-(n) + G_-(n-1) - \alpha\beta F_+(n+1)F_-(n) - G_+(n+1)G_-(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1), \tag{3.4}$$

$$\frac{d}{d\tau}\ln[1 - T(n) - G_+(n) + G_-(n)] = G_+(n+1) - G_+(n) + G_-(n-1) - G_-(n) - \alpha\beta F_+(n+1)F_-(n) - G_+(n+1)G_-(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1). \tag{3.5}$$

The present concise form (3.1)–(3.5) of evolution equations has been acquired due to the use of some lowest local conservation laws dictated by the matrix structures (2.2) and (2.3) of the spectral and evolution matrices $M(n|z)$ and $B(n|z)$. Evidently, each of three last Eqs. (3.3)–(3.5) has the form of a local conservation law, while the sum of first two Eqs. (3.1) and (3.2) yields one more local conservation law. To say plainly we have exploited the system integrability in order to simplify its notation.

Anyway, according to the very method of their construction the obtained Eqs. (3.1)–(3.5) are said to possess the zero-curvature representation (2.1) with the spectral and evolution matrices $M(n|z)$ and $B(n|z)$ given by formulas (2.2) and (2.3), respectively, where the constraint (2.4) imposed onto the fitting parameters α and β as well as the expressions (2.5)–(2.10) for the constituent parts of evolution matrix $B(n|z)$ have been taken into account. This property proves to be the key indication on an integrability [24] of the system under consideration (3.1)–(3.5).

The given system (3.1)–(3.5) can be treated as the first system from an infinite hierarchy associated with the adopted form (2.2) of a spectral matrix and induced by an infinite set of evolution matrices characterized by the properly chosen dependencies on spectral parameter z. This assertion is in lines with the existence of an infinite hierarchy of local conservation laws, whose densities are determined exclusively by the structure of spectral matrix $M(n|z)$. Except of already announced densities $\ln[1 - T(n) + G_+(n) - G_-(n)]$, $\ln[1 + T(n) - G_+(n) - G_-(n)]$, $\ln[1 - T(n) - G_+(n) + G_-(n)]$, $\ln[F_+(n)F_-(n)]$ we present here the second lowest local densities

$$\rho_2^-(n) = T(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1), \tag{3.6}$$

$$\rho_2^+(n) = T(n) + \alpha\beta F_+(n+1)F_-(n) + G_+(n+1)G_-(n) \tag{3.7}$$

having been found by the direct technique [19], generalizing that of Konno, Sanuki, Ichikawa and Wadati [27,28] to the case of essentially multicomponent integrable systems.

4. Symmetric parametrization of field variables

The question how to fix the sampling function $c(n)$ is tantamount to the problem of imposing an additional constraint onto the five prototype field variables, so that only four of them to be truly independent. In general, there exists a number of variants in selection one of admissible additional constraints [20]. However, we prefer the way allowing to define the sampling function $c(n)$ through some redundant quantity $q(n|n-1)$ and to exclude both of them simultaneously from further consideration.

The approach assumes the following parametrization

$$F_+(n) = F_+ \exp\left[+x_+(n) - y_+(n) + q(n|n-1)\right], \tag{4.1}$$

$$F_-(n) = F_- \exp[-x_-(n) + y_-(n) - q(n+1|n)], \tag{4.2}$$

$$G_+(n) = 1 - \exp\left[+x_-(n) + y_-(n) - y_+(n)\right]\cosh[x_+(n)], \tag{4.3}$$

$$T(n) = 1 - \exp\left[-y_+(n) + y_-(n)\right]\cosh[x_+(n) + x_-(n)], \tag{4.4}$$

$$G_-(n) = 1 - \exp\left[-x_+(n) - y_+(n) + y_-(n)\right]\cosh[x_-(n)], \tag{4.5}$$

where $\dot{F}_+ = 0 = \dot{F}_-$.

The equations of motion for the new field variables $x_+(n), y_+(n)$ and $x_-(n), y_-(n)$ read as follows

$$\dot{x}_+(n) = G_+(n+1) - G_+(n), \tag{4.6}$$

$$\dot{y}_+(n) = T(n) + \alpha\beta F_+(n+1)F_-(n) + G_+(n+1)G_-(n) - \alpha\beta F_+F_-, \tag{4.7}$$

$$\dot{x}_-(n) = G_-(n-1) - G_-(n), \tag{4.8}$$

$$\dot{y}_-(n) = T(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1) - \alpha\beta F_+F_-. \tag{4.9}$$

These equations are seen to be essentially selfconsistent. As for the variable $q(n|n-1)$ it serves mainly for the definition of sampling function $c(n)$ by means of equation

$$\dot{q}(n|n-1) = c(n) + \alpha\beta F_+(n)F_-(n-1) + G_+(n)G_-(n-1) - \alpha\beta F_+F_-. \tag{4.10}$$

Nevertheless, namely the proper choice of this definition (4.10) ensures the correct frame of reference for the true field variables $x_+(n), y_+(n)$ and $x_-(n), y_-(n)$ due to the presence of last term $-\alpha\beta F_+F_-$ in the right-hand sides of Eqs. (4.7) and (4.9) for $\dot{y}_+(n)$ and $\dot{y}_-(n)$.

The structure of Eqs. (4.7), (4.9) and (4.10) prompts us to adopt $F_+F_- = 1$ without the loss of generality.

In order to avoid unnecessary complications when dealing with zero-curvature formulation (2.1) of reduced system (4.6)–(4.9) it is reasonable to equalize the redundant variable $q(n|n-1)$ to zero. As a result both the spectral and evolution matrices $M(n|z)$ and $B(n|z)$ will be strictly defined inasmuch as $q(n|n-1)=0$ and

$$c(n) = \alpha\beta F_+F_- - \alpha\beta F_+(n)F_-(n-1) - G_+(n)G_-(n-1). \quad (4.11)$$

The obtained four-wave system (4.6)–(4.9) can be understood as a nonlinear vibrational system given on some two-leg infinite ladder lattice, where indices − and + mark the sites respectively on the left and right legs respectively, while the unit cells are numbered by the running integer n. In view of modern technological capabilities to synthetize a wide variety of low-dimensional regular nanostructures [29–33], there is a good reason to expect that our four-wave system (4.6)–(4.9) could be adopted as a reliable model for the propagation of pulse-like nonlinear oscillations on a properly chosen ladder lattice.

The possibility to regulate the adjustable coupling parameter $\alpha\beta$ without the loss of system integrability assumes the existence of several qualitatively distinct regimes of nonlinear dynamics and the opportunity of switching over between them. The most evident and simple step to corroborate the above statement is to perform the comprehensive linear analysis of the system under study (4.6)–(4.9) with $\alpha\beta$ serving as an arbitrary real governing parameter.

In what follows we restrict ourselves only to the case of linear consideration.

5. Dispersion equation for the low-amplitude excitations and the general principles of its analysis

Assuming the coupling parameter $\alpha\beta$ to be the real one let us obtain the dispersion equation for the low-amplitude excitations in shortened semidiscrete nonlinear system (4.6)–(4.9). In so doing we linearize the system of our interest (4.6)–(4.9) by expanding its left-hand-side terms near the values $x_+(n)=0, y_+(n)=0$ and $x_-(n)=0$, $y_-(n)=0$ and use the standard plane-wave ansätze

$$x_+(n) = x_+\exp(i\varkappa n - i\Omega\tau), \quad (5.1)$$
$$y_+(n) = y_+\exp(i\varkappa n - i\Omega\tau), \quad (5.2)$$
$$x_-(n) = x_-\exp(i\varkappa n - i\Omega\tau), \quad (5.3)$$
$$y_-(n) = y_-\exp(i\varkappa n - i\Omega\tau). \quad (5.4)$$

Then the spectrum of linearized system

$$\dot{x}_+(n)/k \approx y_+(n+1) - y_-(n+1) - x_-(n+1) - y_+(n) + y_-(n) + x_-(n), \quad (5.5)$$

$$\dot{y}_+(n)/k \approx y_+(n) - y_-(n) + \alpha\beta[x_+(n+1) - y_+(n+1) - x_-(n) + y_-(n)], \quad (5.6)$$

$$\dot{x}_-(n)/k \approx y_+(n-1) - y_-(n-1) + x_+(n-1) - y_+(n) + y_-(n) - x_+(n), \quad (5.7)$$

$$\dot{y}_-(n)/k \approx y_+(n) - y_-(n) + \alpha\beta[x_+(n) - y_+(n) - x_-(n-1) + y_-(n-1)], \quad (5.8)$$

will be determined by the following quartic dispersion equation

$$\Omega^4 - 2\alpha\beta\sin(\varkappa)\Omega^3 - 2[1-\cos(\varkappa)]\Omega^2 + 2\alpha\beta[1-2\cos(\varkappa)][1-\cos(\varkappa)]\Omega^2 + 8\alpha\beta\sin(\varkappa)[1-\cos(\varkappa)]\Omega - 4\alpha\beta[1-\cos(\varkappa)]^2 = 0. \quad (5.9)$$

To examine this Eq. (5.9) on the character of its roots it is appropriate to make the substitution

$$\Omega = 2\lambda\sin(\varkappa/2). \quad (5.10)$$

As a consequence we come to the quartic auxiliary dispersion equation

$$\lambda^4 + a(\varkappa|\alpha\beta)\lambda^3 + b(\varkappa|\alpha\beta)\lambda^2 + c(\varkappa|\alpha\beta)\lambda + d(\varkappa|\alpha\beta) = 0, \quad (5.11)$$

where

$$a(\varkappa|\alpha\beta) = -2\alpha\beta\cos(\varkappa/2), \quad (5.12)$$
$$b(\varkappa|\alpha\beta) = -1 + 3\alpha\beta - 4\alpha\beta\cos^2(\varkappa/2), \quad (5.13)$$
$$c(\varkappa|\alpha\beta) = +4\alpha\beta\cos(\varkappa/2), \quad (5.14)$$
$$d(\varkappa|\alpha\beta) = -\alpha\beta. \quad (5.15)$$

Its discriminant [34,35] $\mathscr{D}(\varkappa|\alpha\beta)$ is given by formula [34,36,37]

$$\begin{aligned}\mathscr{D}(\varkappa|\alpha\beta) = {} & a^2(\varkappa|\alpha\beta)b^2(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta) - 4a^3(\varkappa|\alpha\beta)c^3(\varkappa|\alpha\beta) \\ & - 4b^3(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta) + 18a(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta)c^3(\varkappa|\alpha\beta) \\ & - 27c^4(\varkappa|\alpha\beta) + 256d^3(\varkappa|\alpha\beta) \\ & - 4a^2(\varkappa|\alpha\beta)b^3(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) \\ & + 18a^3(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) \\ & + 16b^4(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) \\ & - 80a(\varkappa|\alpha\beta)b^2(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) \\ & - 6a^2(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) \\ & + 144b(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) \\ & - 27a^4(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta) \\ & + 144a^2(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta) \\ & - 128b^2(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta) \\ & - 192a(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta). \end{aligned} \quad (5.16)$$

In the regions of negative discriminant $\mathscr{D}(\varkappa|\alpha\beta) < 0$ the theory of quartic equations [34–36] predicts two real roots and two complex roots. However, in the regions of positive discriminant $\mathscr{D}(\varkappa|\alpha\beta) > 0$ the situation turns out to be ambiguous until we invoke two seminvariants $H(\varkappa|\alpha\beta)$ and $Q(\varkappa|\alpha\beta)$ given by formulas [36,37]

$$H(\varkappa|\alpha\beta) = 8b(\varkappa|\alpha\beta) - 3a^2(\varkappa|\alpha\beta), \quad (5.17)$$
$$Q(\varkappa|\alpha\beta) = 3a^4(\varkappa|\alpha\beta) - 16a^2(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta) + 16a(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta) + 16b^2(\varkappa|\alpha\beta) - 64d(\varkappa|\alpha\beta) \quad (5.18)$$

and determine their signs. Precisely at $\mathscr{D}(\varkappa|\alpha\beta) > 0$ the general theory [34–37] predicts four real roots provided $H(\varkappa|\alpha\beta) < 0$ and $Q(\varkappa|\alpha\beta) > 0$ or four complex roots provided either $H(\varkappa|\alpha\beta) > 0$ or $Q(\varkappa|\alpha\beta) < 0$.

6. Structure of low-amplitude dispersion law as the function of adjustable coupling parameter $\alpha\beta$

In as much as the quantities $\mathcal{D}(\varkappa|\alpha\beta), H(\varkappa|\alpha\beta)$ and $Q(\varkappa|\alpha\beta)$ depend only on two parameters $\varkappa$ and $\alpha\beta$ it is convenient to use the plane of these parameters in order to visualize the regions where signs of $\mathcal{D}(\varkappa|\alpha\beta), H(\varkappa|\alpha\beta)$ and $Q(\varkappa|\alpha\beta)$ remain fixed. Fig. 1 shows such regions found by the computer simulation. Each region is marked by three vertically arranged signs (column of three signs) so that the upper sign is related to $\mathcal{D}(\varkappa|\alpha\beta)$, the middle sign is related to $H(\varkappa|\alpha\beta)$ and the lower sign is related to $Q(\varkappa|\alpha\beta)$. N.B. The caudal-fin like region (although being unlabeled due to the lack of space) is understood as marked by col$(+-+)$ signature.

At $\varkappa = 0$ and $\varkappa = \pm\pi$ the results presented on Fig. 1 permit the direct analytical interpretation based on the simple specific expressions

$$\mathcal{D}(0|\alpha\beta) = 16\alpha\beta(2\alpha\beta - 1)^2[17(\alpha\beta)^3 + 33(\alpha\beta)^2 - 12\alpha\beta - 1], \tag{6.1}$$

$$H(0|\alpha\beta) = -4[(\alpha\beta + 1)^2 + 2(\alpha\beta)^2 + 1], \tag{6.2}$$

$$Q(0|\alpha\beta) = 16[3(\alpha\beta)^4 + 4(\alpha\beta)^3 - 3(\alpha\beta)^2 + 6\alpha\beta + 1] \tag{6.3}$$

and

$$\mathcal{D}(\pm\pi|\alpha\beta) = -16\alpha\beta[(\alpha\beta - 1)^2 + 8(\alpha\beta)^2], \tag{6.4}$$

$$H(\pm\pi|\alpha\beta) = 8(3\alpha\beta - 1), \tag{6.5}$$

$$Q(\pm\pi|\alpha\beta) = 16[(\alpha\beta - 1)^2 + 8(\alpha\beta)^2] \tag{6.6}$$

following from the general ones (5.12)–(5.18).

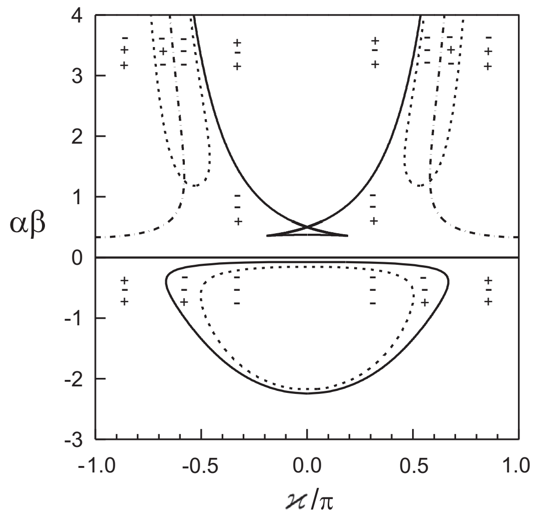

Fig. 1. Subdivision into the regions with fixed signs of $\mathcal{D}(\varkappa|\alpha\beta)$ (upper sign), $H(\varkappa|\alpha\beta)$ (middle sign), $Q(\varkappa|\alpha\beta)$ (lower sign) in the plane of wave vector $\varkappa$ and adjustable coupling parameter $\alpha\beta$. The curves $\mathcal{D}(\varkappa|\alpha\beta) = 0$ are marked by the solid lines. The curves $H(\varkappa|\alpha\beta) = 0$ are marked by the dot-and-dash lines. The curves $Q(\varkappa|\alpha\beta) = 0$ are marked by the dotted lines. The caudal-fin region should be understood as marked by col$(+-+)$ signature. The regions with col$(+-+)$ signature correspond to four stable branches in dispersion law. All other regions correspond to two stable branches in dispersion law.

Thus at $\varkappa = 0$ the formula (6.1) for $\mathcal{D}(0|\alpha\beta)$ ensures that all six roots of equation $\mathcal{D}(0|\alpha\beta) = 0$ must be purely real. To wit we have

$$(\alpha\beta)_1 \simeq -2.2441, \tag{6.7}$$

$$(\alpha\beta)_2 \simeq -0.0703, \tag{6.8}$$

$$(\alpha\beta)_3 = 0, \tag{6.9}$$

$$(\alpha\beta)_4 \simeq 0.3731, \tag{6.10}$$

$$(\alpha\beta)_5 = 0.5, \tag{6.11}$$

$$(\alpha\beta)_6 = 0.5, \tag{6.12}$$

where the roots $(\alpha\beta)_3, (\alpha\beta)_5, (\alpha\beta)_6$ are selfevident, while the roots $(\alpha\beta)_1, (\alpha\beta)_2, (\alpha\beta)_4$ are obtainable from the cubic equation whose discriminant [34,36] is proved to be positive. As for the quantity $H(0|\alpha\beta)$ (formula (6.2)) it is seen to be essentially negative. At last the quartic equation $Q(0|\alpha\beta) = 0$ (see formula (6.3) for $Q(0|\alpha\beta)$) must possess two real negative roots, since its discriminant is negative and $Q(0|\alpha\beta \geqslant 0) > 0$ while the parameter $\alpha\beta$ must be purely real by definition.

The consideration of formulas (6.4)–(6.6) for $\mathcal{D}(\pm\pi|\alpha\beta)$, $H(\pm\pi|\alpha\beta)$ and $Q(\pm\pi|\alpha\beta)$ related to $\varkappa = \pm\pi$ yields $\mathcal{D}(\pm\pi|\alpha\beta > 0) < 0, \mathcal{D}(\pm\pi|\alpha\beta < 0) > 0$, and $H(\pm\pi|\alpha\beta > 1/3) > 0$, $H(\pm\pi|\alpha\beta < 1/3) < 0$, whereas $Q(\pm\pi|\alpha\beta) > 0$ for all real $\alpha\beta$.

Examining the analytical data contained in previous three paragraphs we clearly observe their one-to-one reproductions on the lines $\varkappa = 0$ and $\varkappa = \pm\pi$ of Fig. 1.

According to the commonly accepted graphical treatment of a dispersion law [38–40] we shall be interested with the real-valued solutions $\lambda_j^*(\varkappa|\alpha\beta) = \lambda_j(\varkappa|\alpha\beta)$ of auxiliary dispersion equation (5.11) at the real-valued wave vector $\varkappa$ spanning the first Brillouin zone $-\pi \leqslant \varkappa \leqslant +\pi$. Thus juxtaposing the signatures of all regions pictured on Fig. 1 with the early listed criteria on the character of roots we can readily conclude that the regions marked by col$(+-+)$ signature should produce the four-branch auxiliary dispersion law, while in the other regions the auxiliary dispersion law should be the two-branch one. The same statement concerns also the actual dispersion law, i.e. dispersion law given in terms of cyclic eigenfrequencies $\Omega_j(\varkappa|\alpha\beta) = 2\lambda_j(\varkappa|\alpha\beta)\sin(\varkappa/2)$, except of the merging point $\varkappa = 0$ where $\Omega_j(\varkappa = 0|\alpha\beta) \equiv 0$. Here the integer j denotes the ordinal number of eigenmode of the low-amplitude excitations and serves as the branch number in their dispersion law.

Fig. 2 demonstrates the typical metamorphoses in the actual low-amplitude dispersion law as the adjustable coupling parameter $\alpha\beta$ grows from the values smaller than $(\alpha\beta)_1$ to the values larger than $(\alpha\beta)_5 \equiv (\alpha\beta)_6$. As we have expected the most crucial qualitative rearrangements in the dispersion law are seen to happen when the value of coupling parameter $\alpha\beta$ passes through any root $(\alpha\beta)_k$ of equation $\mathcal{D}(0|\alpha\beta) = 0$ or through the value $(\alpha\beta)_* \simeq 0.3557$ being the ordinate for each of two symmetric cusp points on the line $\mathcal{D}(\varkappa|\alpha\beta) = 0$ (see the caudal-fin region on Fig. 1). In this respect the points $(\alpha\beta)_k$ (with $k = 1, 2, 3, 4, 5, 6$) and $(\alpha\beta)_*$ can be referred to as the critical ones. Meanwhile, when the coupling parameter $\alpha\beta$ varies between any two distinct neighboring critical points, the changes in a structure of dispersion law are proved to be

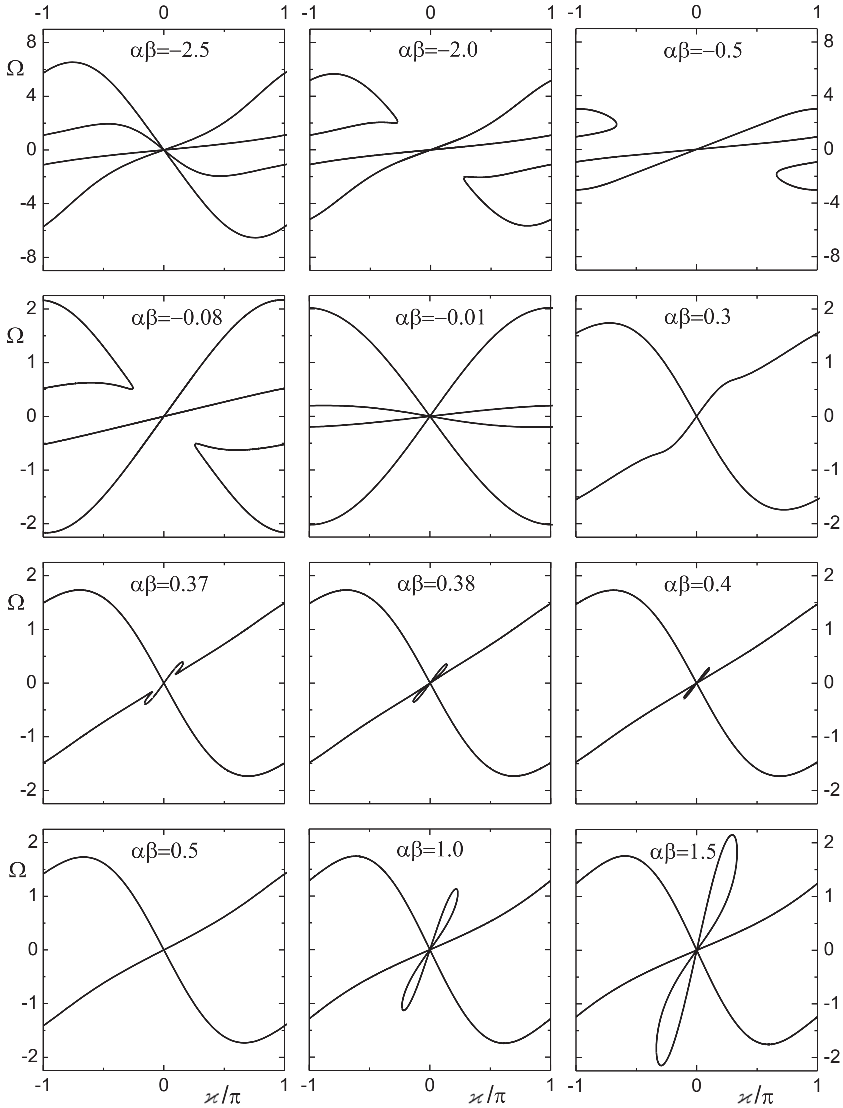

Fig. 2. Real-valued normal cyclic frequencies as a function of wave vector at twelve distinct values of adjustable coupling parameter.

solely quantitative rather than qualitative in complete agreement with the criteria evaluating the character of admissible roots of quartic auxiliary dispersion equation (5.11).

Considering the dispersion curves on each subfigure of Fig. 2 we observe that some of them have the dead-end points with respect to wave vector $\varkappa$. Nonetheless each dead-end point is seen to be shared by two distinct branches. As a consequence the combination of such joint branches can be treated as the unique loop-like branch or the unique folding branch. Here we would like to point out on a certain similarity between the low-amplitude oscillations in our semidiscrete system (4.6)–(4.9) and the oscillations in beam-plasma system where the loop-like structure of dispersion law is known to be rather typical [39,40]. Regarding the linearized version (5.5)–(5.8) of reduced integrable system (4.6)–(4.9) the Fig. 1 prompts us that the loop-like structure of a dispersion law must inevitably emerge once the coupling parameter $\alpha\beta$ enters into one of the following two intervals:

$$(\alpha\beta)_4 < \alpha\beta < (\alpha\beta)_5 \equiv (\alpha\beta)_6, \tag{6.13}$$

$$(\alpha\beta)_5 \equiv (\alpha\beta)_6 < \alpha\beta < \infty, \tag{6.14}$$

where the critical points $(\alpha\beta)_k$ are given by formulas (6.7)–(6.12). As for the folding branch structure of a dispersion law it must be prescribed to the interval

$$(\alpha\beta)_* < \alpha\beta < (\alpha\beta)_4, \tag{6.15}$$

where $(\alpha\beta)_* \simeq 0.3557$ as we already know. The peculiarities concerning the loop-like and folding-like dispersion curves are clearly seen on a respective subfigures of Fig. 2.

It is worth noticing that at large values $|\alpha\beta| \gg 1$ of adjustable coupling parameter $\alpha\beta$ the high amplitude curve (which we shall denote as $\Omega_a(\varkappa|\alpha\beta)$) of a dispersion law can be estimated analytically in rather wide interval

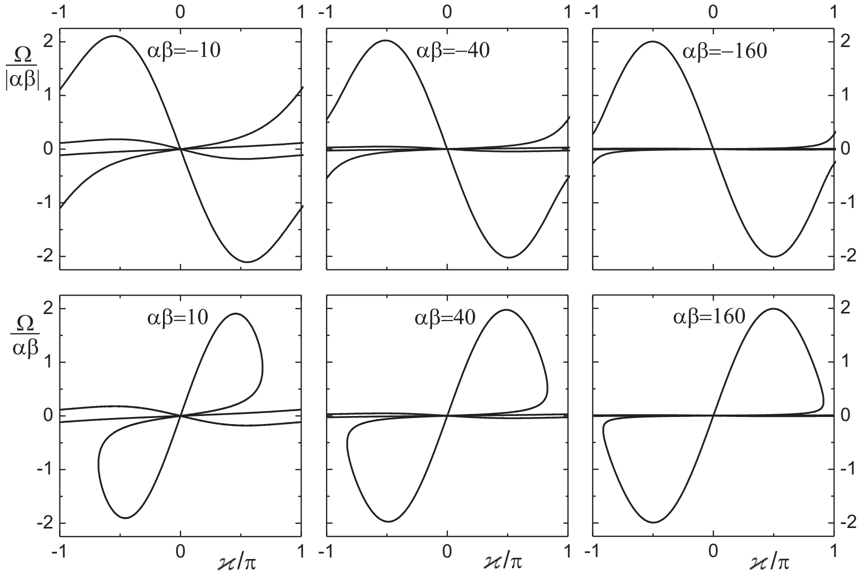

Fig. 3. Real-valued normal cyclic frequencies as a function of wave vector at large negative (upper row) and large positive (lower row) values of adjustable coupling parameter.

of wave vectors. Indeed the calculation based on the auxiliary dispersion equation (5.11) and on the relationship (5.10) between λ and Ω yields the approximate formula

$$\Omega_a(\varkappa|\alpha\beta) \approx 2\alpha\beta\sin(\varkappa), \tag{6.16}$$

with the accuracy controlled by the inequality

$$4|\alpha\beta|\cos^2(\varkappa/2) \gg 3. \tag{6.17}$$

Fig. 3 clearly illustrates that the wave vector interval of accurate reproduction of high amplitude curve by sinusoid (6.16) is increased as the strength $|\alpha\beta|$ of adjustable coupling parameter $\alpha\beta$ grows.

From the standpoint of stability analysis any interval of wave vectors supporting four real-valued branches of quartic dispersion relation (5.9) should be treated as an interval of stability while any interval of wave vectors supporting two real-valued branches of quartic dispersion relation (5.9) should be considered as an interval of instability. Here due to the spatial discreteness of our linear system (5.5)–(5.8) it seems impossible to apply the Sturrock rules [38,39] and to qualify each particular instability either as convective or absolute one.

For us however it is more important the mere manifestation of instability rather than the strict identification of its type. Here we bear in mind the effect of spontaneous symmetry breaking as it is understood in the theory of fields [21,22]. Having been applied to our consideration it requires the existence of two stable branches (instead of four ones) in low-amplitude dispersion law taken at $\varkappa = 0$. According to our calculations the spontaneous symmetry breaking may occur provided the adjustable coupling parameter $\alpha\beta$ belongs to one of two windows

$$(\alpha\beta)_1 < \alpha\beta < (\alpha\beta)_2, \tag{6.18}$$
$$(\alpha\beta)_3 < \alpha\beta < (\alpha\beta)_4, \tag{6.19}$$

or is equal to the value $(\alpha\beta)_5 \equiv (\alpha\beta)_6$. In this situation we do not exclude the possibility of introducing the shifted field variables taking into account the broken symmetry in the relevant nonlinear system (4.6)–(4.9).

In any event the predisposition to instability is not the only property of considered linearized system (5.5)–(5.8), since it can be canceled by the proper choice of adjustable coupling parameter $\alpha\beta$. Precisely at

$$-\infty < \alpha\beta < (\alpha\beta)_1 \tag{6.20}$$

or at

$$(\alpha\beta)_2 < \alpha\beta < (\alpha\beta)_3 \tag{6.21}$$

all four branches of low-amplitude excitations are stable for all wave vectors $-\pi \leqslant \varkappa \leqslant +\pi$ thus ensuring the good background for the stable solutions of semidiscrete nonlinear system of interest (4.6)–(4.9) already in terms of early adopted field variables $x_+(n), y_+(n)$ and $x_-(n), y_-(n)$.

7. Alternative version of the four-wave system as the generalization of nonlinear self-dual network model

In this section we will try to elucidate the feasible physical meaning of reduced four-wave system (4.6)–(4.9). In so doing we abandon its symmetric parametrization (4.1)–(4.5) bearing in mind the possibility to use another closed set of true field variables. For instance we can rely upon the variables $F_+(n+1)F_-(n), G_+(n), G_-(n), T(n)$ which are completely insensitive to the choice of the sampling function $c(n)$.

The later case might be essentially improved by abandoning the awkward composite variable $F_+(n+1)F_-(n)$ in favor of variable $w(n)$ that arises from the constraint

$$\frac{d}{d\tau}\ln\frac{F_+(n)F_-(n)}{[1-T(n)]^2 - [G_+(n)-G_-(n)]^2} = 0, \tag{7.1}$$

due to the following parametrization

$$F_+(n) = [1 - T(n) - G_+(n) + G_-(n)]F_+ \exp[+w(n)], \tag{7.2}$$

$$F_-(n) = [1 - T(n) + G_+(n) - G_-(n)]F_- \exp[-w(n)]. \tag{7.3}$$

Here $\dot{F}_+ = 0 = \dot{F}_-$. The adopted constraint (7.1) fixates the sampling function $c(n)$ by the expression

$$c(n) = c + G_+(n) + G_-(n-1), \tag{7.4}$$

where the summation parameter c could be thought as an arbitrary function of time τ.

Taking into account the above findings (7.2)–(7.4) let us rearrange the general integrable system (3.1)–(3.5) into the following reduced form

$$\frac{d}{d\tau}\ln\frac{1-T(n)+G_+(n)-G_-(n)}{1+T(n)-G_+(n)-G_-(n)} = 2G_-(n) - 2G_-(n-1), \tag{7.5}$$

$$\frac{d}{d\tau}\ln\frac{1-T(n)-G_+(n)+G_-(n)}{1+T(n)-G_+(n)-G_-(n)} = 2G_+(n+1) - 2G_+(n), \tag{7.6}$$

$$\begin{aligned}&\frac{d}{d\tau}\ln\left\{[1-T(n)]^2 - [G_+(n)-G_-(n)]^2\right\}\\ &= 2\alpha\beta F_+(n)F_-(n-1) - 2\alpha\beta F_+(n+1)F_-(n)\\ &\quad + 2G_+(n)G_-(n-1) - 2G_+(n+1)G_-(n),\end{aligned} \tag{7.7}$$

$$\dot{w}(n) = G_+(n) + G_-(n) - T(n), \tag{7.8}$$

where the expressions for $F_+(n)$ and $F_-(n)$ are given by the adopted parametrization formulas (7.2) and (7.3) respectively, while the quantities $G_+(n), G_-(n)$ and $T(n), w(n)$ should be treated as the field variables. In obtained Eqs. (7.5)–(7.8) we have assumed the parameter c to be of zero value insofar as even at $c \neq 0$ the system equations can be readily transformed into their present form (7.5)–(7.8) by a proper coordinate independent shift of field variable $w(n)$.

Similarly to the case with symmetric parametrization we can adopt $F_+F_- = 1$ in the third Eq. (7.7) without the loss of generality.

At $\alpha\beta = 0$ the last Eq. (7.8) becomes redundant inasmuch as the first three ones (7.5)–(7.7) do not contain the variable $w(n)$ at all. Simultaneously the third Eq. (7.7) loses its independence (i.e. it can be obtained by the proper combination of the first two ones (7.5) and (7.6)). As a consequence the so-called natural constraint

$$\begin{aligned}&D(n)\left[1 + T^2(n) - G_+^2(n) - G_-^2(n)\right]\\ &= E(n)[T(n) - G_+(n)G_-(n)]\end{aligned} \tag{7.9}$$

turns out to be valid, where $D(n)$ and $E(n)$ are some time-independent parameters. Assuming the uniformity of space these parameters must be taken also as independent on the space variable: $D(n) = D$ and $E(n) = E$. In the case when quantities $G_+(n), G_-(n)$ and $T(n)$ are adopted as vanishing at both spatial infinities $|n| \to \infty$ the single constructive variant to satisfy the natural constraint (7.9) is to admit that $D \equiv 0$ and $E \neq 0$. As a result we have

$$T(n) = G_+(n)G_-(n). \tag{7.10}$$

Thus at $\alpha\beta = 0$ only two equations

$$\frac{\dot{G}_+(n)}{1 - G_+^2(n)} = G_-(n) - G_-(n-1), \tag{7.11}$$

$$\frac{\dot{G}_-(n)}{1 - G_-^2(n)} = G_+(n+1) - G_+(n) \tag{7.12}$$

remain significant.

Provided $G_+(n)$ being identified with the dimensionless current $I(n)$ and $G_-(n)$ with the dimensionless voltage $V(n)$, these two coupled equations (7.11) and (7.12) manifest themselves as the nonlinear ladder network system of self-dual type consisting of current-dependent inductors and voltage-dependent capacitors with nth inductance $L(I(n))$ and nth capacitance $C(V(n))$ characterizing by the following nonlinearities

$$L(I(n)) = \frac{\text{artanh}[I(n)]}{I(n)} \tag{7.13}$$

and

$$C(V(n)) = \frac{\text{artanh}[V(n)]}{V(n)}. \tag{7.14}$$

The equivalent electric circuit scheme of this truncated system (7.11) and (7.12) happens to be formally the same as that constructed by Hirota and Suzuki [41] or that analytically investigated by Hirota [42] or at last that considered numerically by Daikoku, Mizushima, and Tamama [43] and analytically by Hirota and Satsuma [44]. Having been based upon the same electric scheme the listed classical models are distinguished by the particular dependencies of inductance $L(I(n))$ and capacitance $C(V(n))$ on their arguments thus giving rise to distinct physical backgrounds. For example, a proper choice of these dependencies allow one to model the systems [43,44] originated from the Volterra competition equations [45]. The self-duality of obtained truncated system (7.11) and (7.12) is closely linked with the self-duality of Hirota one [42] and it is supported by the one-to-one correspondence between these systems through the formal substitutions $G_+(n) = iI(n), G_-(n) = iV(n)$ and the inversion of time $\tau \to -\tau$.

At an arbitrary value of adjustable coupling parameter $\alpha\beta$ the integrable four-wave lattice system (7.5)–(7.8) can be interpreted as an nontrivial extension of nonlinear self-dual network system specified by two coupled lattice subsystems of principally distinct origins. In order to substantiate this statement it is reasonable to introduce the modified field variables $g_+(n), t(n), g_-(n)$ instead of the former ones $G_+(n), T(n), G_-(n)$. The respective transformation formulas are as follows

$$G_+(n) = g_+(n) + [1 - g_+(n)]t(n), \tag{7.15}$$

$$T(n) = [1 - g_+(n)g_-(n)]t(n) + g_+(n)g_-(n), \tag{7.16}$$

$$G_-(n) = g_-(n) + [1 - g_-(n)]t(n). \tag{7.17}$$

These formulas (7.15)–(7.17) in particular yield

$$1 - T(n) + G_+(n) - G_-(n) = [1 + g_+(n)][1 - t(n)][1 - g_-(n)], \tag{7.18}$$

$$1 + T(n) - G_+(n) - G_-(n) = [1 - g_+(n)][1 - t(n)][1 - g_-(n)], \tag{7.19}$$

$$1 - T(n) - G_+(n) + G_-(n) = [1 - g_+(n)][1 - t(n)][1 + g_-(n)] \tag{7.20}$$

and

$$F_+(n) = [1 - g_+(n)][1 - t(n)][1 + g_-(n)]F_+ \exp[+w(n)], \tag{7.21}$$

$$F_-(n) = [1 + g_+(n)][1 - t(n)][1 - g_-(n)]F_- \exp[-w(n)]. \tag{7.22}$$

As a result the integrable system of our interest (7.5)–(7.8) is presentable as two coupled nonlinear lattice subsystems

$$\frac{\dot{g}_+(n)}{1-g_+^2(n)} = G_-(n) - G_-(n-1), \tag{7.23}$$

$$\frac{\dot{g}_-(n)}{1-g_-^2(n)} = G_+(n+1) - G_+(n) \tag{7.24}$$

and

$$\frac{\dot{t}(n)}{1-t(n)} = \alpha\beta F_+(n+1)F_-(n) - \alpha\beta F_+(n)F_-(n-1)$$
$$+ G_+(n+1)G_-(n) - G_+(n)G_-(n-1)$$
$$- G_+(n+1)g_-(n) + g_+(n)G_-(n-1)$$
$$+ G_+(n)g_-(n) - g_+(n)G_-(n), \tag{7.25}$$

$$\dot{w}(n) = G_+(n) + G_-(n) - T(n) \tag{7.26}$$

where the variables $g_+(n)$ and $g_-(n)$ are responsible for a sort of nonlinear self-dual network subsystem (Eqs. (7.23) and (7.24)), while the variables $t(n)$ and $w(n)$ are related to some nonlinear vibrational lattice subsystem (Eqs. (7.25) and (7.26)) forced by the self-dual one.

The consideration of low-amplitude excitations in the above reduced system (7.23)–(7.26) gives rise to the same quartic dispersion equation as that (5.9) inherent in the symmetrically reduced system (4.6)–(4.9). Thus the results concerning the low-amplitude spectrum (Sections 5 and 6) are applicable to both of proposed reductions (4.6)–(4.9) and (7.23)–(7.26) on an equal footing.

8. Conclusion

Considering the nondegenerate 3×3 matrices we have selected a properly truncated Laurent-type form of the spectral and evolution matrices to be mutually consistent in the framework of matrix-valued semidiscrete zero-curvature equation and have isolated early unknown basic integrable nonlinear evolution system on quasionedimensional lattice.

When choosing the fixation of sampling function one can come to distinct reductions of basic semidiscrete system and as a consequence to distinct parametrizations of prototype field amplitudes. In this paper we have presented two possible paramertizations, each one supporting a sort of reduced four-wave integrable system.

The integrable system (either the reduced or the basic one) is characterized by two parameters (one of which we have normalized to unity) responsible for the interfield couplings of principally distinct origins. The variation of adjustable coupling parameters should lead to several regimes in system behavior with nontrivial dynamics. This statement find its natural confirmation already on the stage of low-amplitude plane-wave excitations whose dispersion law turns out to be essentially dependent on the magnitude and sign of adjustable coupling parameter provided other coupling parameter is fixed. Thus depending on the value of adjustable coupling parameter the reduced four-wave system can exhibit at least six qualitatively distinct realizations of its low-amplitude dispersion law distinguished by the number of stable branches as well as by the graphic (smooth, folding-like or loop-like) structure of available branches.

In some windows of adjustable coupling parameter we observe the clear resemblance between the obtained dispersion curves and those typical of the beam-plasma oscillations in hydrodynamic plasma [39,40].

On the other hand there are all reasons to believe that the intervals of adjustable coupling parameter ensuring only two stable branches of low-amplitude dispersion law in the long wave limit $\varkappa = 0$ may predetermine the windows of spontaneous symmetry breaking in the nonlinear system under study.

There are also windows of adjustable coupling parameter where all branches of low-amplitude oscillations remain stable within the whole Brillouin zone.

Inspecting the list of already known semidiscrete nonlinear integrable systems one can reveal a considerable structural analogy between the alternative version of four wave nonlinear ladder system (7.23)–(7.26) and the two-wave nonlinear ladder network system considered in details by Hirota [42]. In this connection the perspective of applicability for the proposed four-wave ladder system in either of its two considered forms (4.6)–(4.9) or (7.23)–(7.26) appears to be rather promising due in parts to its potentially rich dynamics and practically inexhaustible constructive variativity of nonlinear electric circuits, in particular the nonlinear electric transmission lines with ladder-like network configurations [8].

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