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The N loop soliton solution of the Vakhnenko equation

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Abstract. An exact N loop soliton solution to the Vakhnenko equation (VE) is found, where $N \geqslant 2$ is an arbitrary positive integer. The key step in finding this solution is to transform the independent variables in the equation. This leads to a transformed equation for which it is straightforward to find an exact explicit N-soliton solution by using Hirota's method. The exact N loop soliton solution to the VE is then found in implicit form by means of a transformation back to the original independent variables. The shifts that occur when the solitons interact are found. The general results and details of the interaction between solitons are illustrated for the case $N = 3$.

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1. Introduction

In [1] we discussed the two loop soliton solution of the nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0 \tag{1.1}$$

hereafter referred to as the Vakhnenko equation (VE). The key step in finding this solution was to transform the independent variables in (1.1). This led to an equation that can be expressed in bilinear form in terms of the Hirota D operator [2]. It was straightforward to find the exact explicit 2-soliton solution to this equation by use of Hirota's method. The exact two loop soliton solution to the VE was then found in implicit form by means of a transformation back to the original independent variables.

The aim of the present paper is to discuss the N loop soliton solution of (1.1) for a general positive integer $N \geqslant 2$, and to illustrate our general results for the particular case $N = 3$.

In section 2 we summarize the transformation of the VE into an equation in bilinear form. In section 3 we give details of the N loop soliton solution to the VE. In section 4 we discuss aspects of this solution. In section 5 we illustrate our results by considering in some detail the case $N = 3$. A proof of the 'N-soliton condition' is given in the appendix.

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2. Transformation of the VE

Here we summarize the transformation of the VE into an equation in bilinear form as described in [1].

We introduce new independent variables X, T defined by

$$x = \theta(X,T) := T + \int_{-\infty}^{X} U(X',T)\,\mathrm{d}X' + x_0, \qquad t = X, \tag{2.1}$$

where $u(x,t) = U(X,T)$, and x_0 is a constant. We also introduce W defined by

$$W_X = U \tag{2.2}$$

and assume that, as $X \to -\infty$, the derivatives of W vanish and W tends to a constant. In [1] it was shown that (1.1) transforms into

$$W_{XXT} + W_X W_T + W_X = 0. \tag{2.3}$$

Then by taking

$$W = 6(\ln f)_X \tag{2.4}$$

(2.3) may be written as the bilinear equation

$$F(D_X, D_T) f \cdot f = 0, \tag{2.5}$$

where

$$F(D_X, D_T) := D_T D_X^3 + D_X^2. \tag{2.6}$$

The solution procedure for the VE is as follows. We solve (2.5) for f by using Hirota's method [2] and hence find $W(X,T)$ and $U(X,T)$ by using (2.4) and (2.2), respectively. The solution to the VE is then given in parametric form by

$$u(x,t) = U(t,T), \qquad x = \theta(t,T), \tag{2.7}$$

where

$$\theta(X,T) = T + W(X,T) + x_0. \tag{2.8}$$

3. The N loop soliton solution of the VE

Using Hirota's method [2] the solution to (2.5) corresponding to N solitons is given by

$$f = \sum_{\mu=0,1} \exp\left[2\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij}\right)\right], \qquad \text{where} \quad \eta_i = k_i X - \omega_i T + \alpha_i, \tag{3.1}$$

$$b_{ij}^2 = -\frac{F[2(k_i - k_j), -2(\omega_i - \omega_j)]}{F[2(k_i + k_j), -2(\omega_i + \omega_j)]}, \tag{3.2}$$

and k_i, ω_i and α_i are constants. In (3.1) $\sum_{\mu=0,1}$ means the summation over all possible combinations of $\mu_1 = 0$ or 1, $\mu_2 = 0$ or 1, $\ldots$, $\mu_N = 0$ or 1, and $\sum_{i<j}^{(N)}$ means the summation over all possible combinations of N elements under the condition $i < j$.

(3.1) is a solution to (2.5) provided the 'N-soliton condition' holds [2]. In the appendix we discuss this condition with F given by (2.6).

With F given by (2.6) the dispersion relations $F(2k_i, -2\omega_i) = 0$ $(i = 1, \ldots, N)$ give $\omega_i = 1/4k_i$ and then

$$\eta_i = k_i(X - c_i T) + \alpha_i \qquad \text{with} \quad c_i = 1/4k_i^2. \tag{3.3}$$

Also, without loss of generality, we may take $k_1 < \cdots < k_N$ and then

$$b_{ij} = \frac{k_j - k_i}{k_i + k_j}\sqrt{\frac{k_i^2 + k_j^2 - k_i k_j}{k_i^2 + k_j^2 + k_i k_j}}, \qquad \text{where} \quad i < j, \tag{3.4}$$

so that $0 < b_{ij} < 1$.

In principle, substitution of (3.1) into (2.4) gives $W(X, T)$. However, following Moloney and Hodnett [3], it is more convenient to express f in the form

$$f = h_i + \hat{h}_i e^{2\eta_i} \tag{3.5}$$

for a given i with $1 \leqslant i \leqslant N$, where

$$h_i = \sum_{\mu=0,1} \exp\left[2\left(\sum_{\substack{r=1\\(r\neq i)}}^{N} \mu_r\eta_r + \sum_{\substack{r<s\\(r\neq i, s\neq i)}}^{(N)} \mu_r\mu_s \ln b_{rs}\right)\right], \tag{3.6}$$

$$\hat{h}_i = \sum_{\mu=0,1} \exp\left[2\left(\sum_{\substack{r=1\\(r\neq i)}}^{N} \mu_r\eta_r + \sum_{\substack{r<s\\(r\neq i, s\neq i)}}^{(N)} \mu_r\mu_s \ln b_{rs} + \sum_{r=1}^{i-1} \mu_r \ln b_{ri} + \sum_{r=i+1}^{N} \mu_r \ln b_{ir}\right)\right], \tag{3.7}$$

and then we may write $W(X, T)$ in the form

$$W = \sum_{i=1}^{N} W_i, \qquad \text{where} \quad W_i = 6k_i(1 + \tanh g_i) \quad \text{and} \quad g_i(X, T) = \eta_i + \tfrac{1}{2} \ln\left[\frac{\hat{h}_i}{h_i}\right]. \tag{3.8}$$

It follows that U may be written

$$U = \sum_{i=1}^{N} U_i, \qquad \text{where} \quad U_i = 6k_i \frac{\partial g_i}{\partial X} \operatorname{sech}^2 g_i. \tag{3.9}$$

The N loop soliton solution to the VE is given by (2.7) and (2.8) with (3.8) and (3.9).

4. Discussion of the *N* loop soliton solution

We now interpret the N loop soliton solution found in section 3 in terms of individual loop solitons.

First it is instructive to consider what happens in X–T space. From (3.8) and (3.9) and the fact that $c_1 > \cdots > c_N$ we deduce the following behaviour: with $X - c_i T$ fixed and $T \to +\infty$,

$$U_i \sim \begin{cases} 6k_1^2 \operatorname{sech}^2 \eta_1, & \text{if} \quad i = 1, \\ 6k_i^2 \operatorname{sech}^2\left(\eta_i + \sum_{r=1}^{i-1} \ln b_{ri}\right), & \text{if} \quad 2 \leqslant i \leqslant N; \end{cases} \tag{4.1}$$

with $X - c_i T$ fixed and $T \to +\infty$,

$$U_i \sim \begin{cases} 6k_i^2 \operatorname{sech}^2\left(\eta_i + \sum_{r=i+1}^{N} \ln b_{ir}\right), & \text{if} \quad 1 \leqslant i \leqslant N-1, \\ 6k_N^2 \operatorname{sech}^2 \eta_N, & \text{if} \quad i = N. \end{cases} \tag{4.2}$$

Hence it is apparent that, in the limits $T \to \pm\infty$, each U_i may be identified as an individual soliton moving with speed c_i in the positive X direction. Smaller solitons overtake larger ones.

The shifts, Δ_i, of the solitons in the positive X direction due to the interactions between the N solitons are given by

$$\Delta_1 = -\frac{1}{k_1} \sum_{r=2}^{N} \ln b_{1r}, \tag{4.3}$$

$$\Delta_i = \frac{1}{k_i}\left(\sum_{r=1}^{i-1} \ln b_{ri} - \sum_{r=i+1}^{N} \ln b_{ir}\right), \qquad 2 \leqslant i \leqslant N-1, \tag{4.4}$$

$$\Delta_N = \frac{1}{k_N} \sum_{r=1}^{N-1} \ln b_{rN}. \tag{4.5}$$

Since the 'mass' of each soliton is given by $\int_{-\infty}^{\infty} U_i \, \mathrm{d}X = 12k_i$, where we have used (3.9), and the shifts satisfy

$$\sum_{i=1}^{N} k_i \Delta_i = 0, \tag{4.6}$$

'momentum' is conserved.

Now let us consider what happens in x–t space. From (2.8) with $v_i = 1/c_i$ we have

$$x - v_i t = -v_i(X - c_i T) + W(X, T) + x_0. \tag{4.7}$$

Note that in (4.1) and (4.2) taking the limits $T \to \pm\infty$ with $X - c_i T$ fixed is equivalent to taking the limits $X \to \pm\infty$ with $X - c_i T$ fixed; also note that $X = t$ from (2.1). Accordingly from (4.1), (4.2) and (4.7), with a given i, we see that in the limits $t \to \pm\infty$ with $X - c_i T$ fixed, $U_i(X, T)$ and $x - v_i t$ are related by the parameter $X - c_i T$. It follows that in the limits $t \to \pm\infty$, u_i may be identified as an individual loop soliton moving with speed v_i in the positive x direction, where $u_i(x, t) = U_i(X, T)$. As $v_1 < \cdots < v_N$, larger loop solitons overtake smaller ones.

In order to calculate the shifts, δ_i, of the loop solitons u_i in the positive x direction due to the interactions between the N loop solitons, we need the following results: from (4.1), as $T \to -\infty$, $U_i \to U_{i\,\max} = 6k_i^2$ where

$$X - c_i T = \begin{cases} -\dfrac{\alpha_1}{k_1}, & \text{for} \quad i = 1 \text{ and then } W \to 6k_1, \\ -\dfrac{\alpha_i}{k_i} - \dfrac{1}{k_i}\displaystyle\sum_{r=1}^{i-1} \ln b_{ri}, & \text{for} \quad 2 \leqslant i \leqslant N \text{ and then } W \to 6k_i + \sum_{r=1}^{i-1} 12k_r; \end{cases}$$

from (4.2), as $T \to \infty$, $U_i \to U_{i\,\max} = 6k_i^2$ where

$$X - c_i T = \begin{cases} -\dfrac{\alpha_i}{k_i} - \dfrac{1}{k_i}\displaystyle\sum_{r=i+1}^{N} \ln b_{ir}, \\ \qquad \text{for} \quad 1 \leqslant i \leqslant N-1 \text{ and then } W \to 6k_i + \sum_{r=i+1}^{N} 12k_r. \\ -\dfrac{\alpha_N}{k_N}, \qquad \text{for} \quad i = N \text{ and then } W \to 6k_N. \end{cases}$$

Using these results in (4.7) gives

$$\delta_1 = \sum_{r=2}^{N} (4k_1 \ln b_{1r} + 12k_r), \tag{4.8}$$

$$\delta_i = \sum_{r=i+1}^{N} (4k_i \ln b_{ir} + 12k_r) - \sum_{r=1}^{i-1} (4k_i \ln b_{ri} + 12k_r), \qquad 2 \leqslant i \leqslant N-1, \tag{4.9}$$

$$\delta_N = -\sum_{r=1}^{N-1} (4k_N \ln b_{rN} + 12k_r). \tag{4.10}$$

Finally we note that, for the interactions to be centred at $X = 0$ and $T = 0$ in X–T space, we require

$$\alpha_1 = -\tfrac{1}{2} \sum_{r=2}^{N} \ln b_{1r}, \tag{4.11}$$

$$\alpha_i = -\tfrac{1}{2} \left(\sum_{r=1}^{i-1} \ln b_{ri} + \sum_{r=i+1}^{N} \ln b_{ir} \right), \qquad 2 \leqslant i \leqslant N-1, \tag{4.12}$$

$$\alpha_N = -\tfrac{1}{2} \sum_{r=1}^{N-1} \ln b_{rN}, \tag{4.13}$$

and then, for the interactions to be centred at $x = 0$ and $t = 0$ in x–t space, we require

$$x_0 = -6\sum_{r=1}^{N} k_r. \tag{4.14}$$

5. Example: $N = 3$

The case $N = 2$ has been discussed in detail in [1]. Here we consider the case $N = 3$.

For $N = 3$, (3.1) gives

$$f = 1 + e^{2\eta_1} + e^{2\eta_2} + e^{2\eta_3} + b_{12}^2 e^{2(\eta_1+\eta_2)} + b_{23}^2 e^{2(\eta_2+\eta_3)} + b_{13}^2 e^{2(\eta_1+\eta_3)} + b_{12}^2 b_{23}^2 b_{13}^2 e^{2(\eta_1+\eta_2+\eta_3)} \tag{5.1}$$

so that, for example, (3.8) gives

$$W_1 = 6k_1(1 + \tanh g_1), \tag{5.2}$$

where

$$g_1(X, T) = \eta_1 + \frac{1}{2}\ln\left[\frac{1 + b_{12}^2 e^{2\eta_2} + b_{13}^2 e^{2\eta_3} + b_{12}^2 b_{23}^2 b_{13}^2 e^{2(\eta_2+\eta_3)}}{1 + e^{2\eta_2} + e^{2\eta_3} + b_{23}^2 e^{2(\eta_2+\eta_3)}}\right]. \tag{5.3}$$

Also (4.8)–(4.10) give

$$\delta_1 = 4k_1(\ln b_{12} + \ln b_{13}) + 12(k_2 + k_3), \tag{5.4}$$

$$\delta_2 = 4k_2(\ln b_{23} - \ln b_{12}) + 12(k_3 - k_1), \tag{5.5}$$

$$\delta_3 = -4k_3(\ln b_{13} + \ln b_{23}) - 12(k_1 + k_2). \tag{5.6}$$

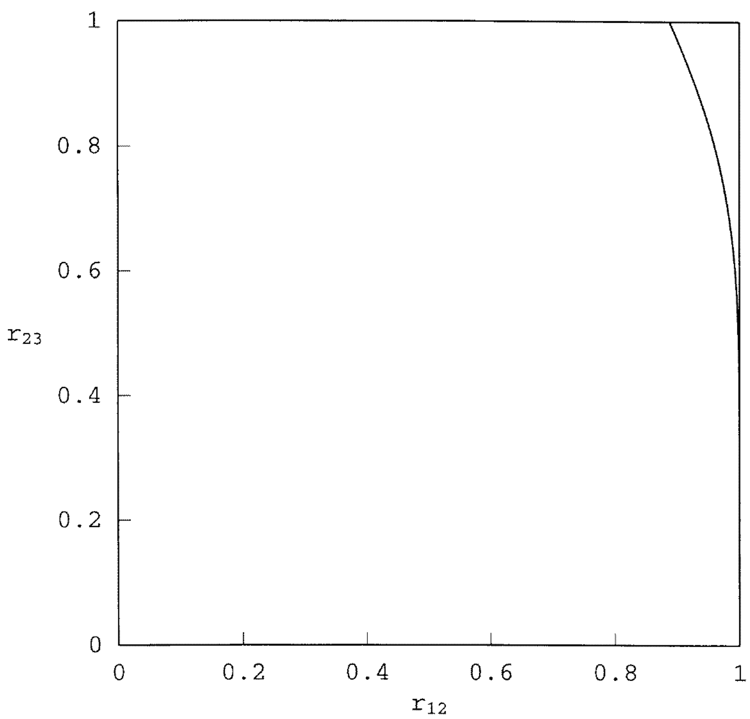

Figure 1. δ_1 as given by (5.4) is positive in the larger region and negative in the smaller region.

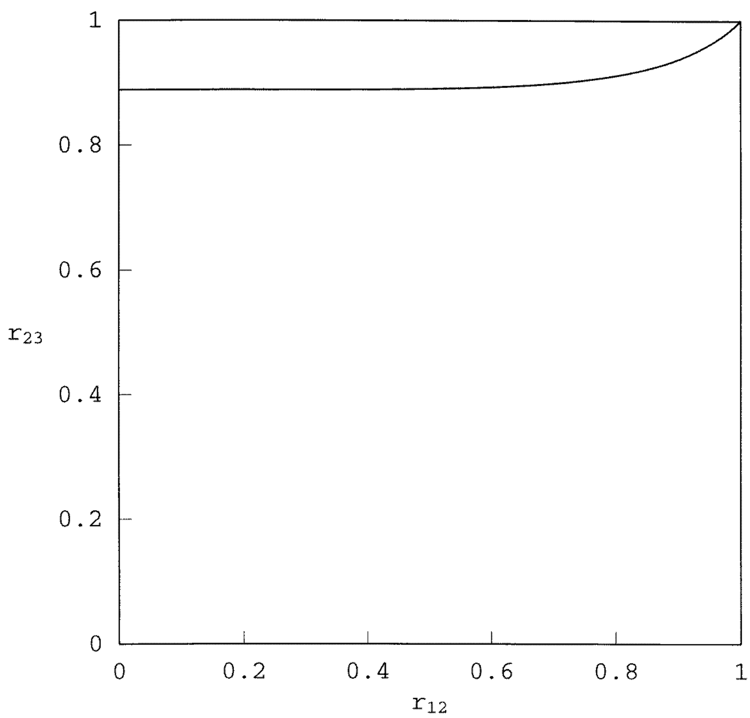

Figure 2. δ_2 as given by (5.5) is positive in the larger region and negative in the smaller region.

It is found that $\delta_3 > 0$ so that the largest loop soliton is always shifted forwards by the interaction with the other two loop solitons. However, δ_1 and δ_2 may be positive or negative depending on the values of the ratios $r_{12} := k_1/k_2$ and $r_{23} := k_2/k_3$. (Note that $0 < r_{12} < 1$ and $0 < r_{23} < 1$.) This is illustrated in figures 1 and 2, respectively. It is clear that it is possible to choose k_1, k_2 and k_3 such that δ_1, δ_2 and δ_3 are all positive. At first sight it might seem that this contradicts conservation of 'momentum'. However, as pointed out in [1], the 'mass' of each loop soliton is zero, and 'momentum' is conserved whatever the values of δ_1, δ_2 and δ_3.

It is of interest to investigate the nature of the interactions between the three loop solitons. Here we consider the case where all three solitons arrive at $x = 0$ at time $t = 0$. Accordingly, from (4.11)–(4.14), we take

$$\alpha_1 = -\tfrac{1}{2}(\ln b_{12} + \ln b_{13}), \qquad \alpha_2 = -\tfrac{1}{2}(\ln b_{12} + \ln b_{23}), \qquad \alpha_3 = -\tfrac{1}{2}(\ln b_{13} + \ln b_{23}), \tag{5.7}$$

and

$$x_0 = -6(k_1 + k_2 + k_3). \tag{5.8}$$

Before discussing the interaction process it is helpful to recall some results for the two loop soliton solution given in [1]. It was shown that there are three characteristic types of behaviour during the interaction process and that these depend on the value of the ratio r_{12} as follows:

(1) for $0.759\,68 < r_{12} < 1$, the two loops exchange their amplitudes during the interaction but never overlap;
(2) for $0.556\,76 < r_{12} < 0.759\,68$, the two loops exchange their amplitudes during the interaction and, for part of the interaction, the loops overlap;

(3) for $0 < r_{12} < 0.556\,76$, the larger loop catches up the smaller loop which then travels clockwise around the larger loop before being ejected behind the larger loop.

For the three loop soliton solution, the behaviour during the interaction process clearly depends on both ratios r_{12} and r_{23}. Similar types of behaviour to the above can be observed for the interaction between loops 1 and 2, and between loops 2 and 3, where we have denoted the loop corresponding to k_i as loop i. Below we consider three illustrative examples. For each example we present a figure in which u is plotted against $x - v_2 t$ at several equally spaced values of t, and give the values of δ_i $(i = 1, 2, 3)$ as calculated from (5.4)–(5.6).

First, consider the case where $k_1 = 0.36$, $k_2 = 0.9$ and $k_3 = 1$ so that $r_{12} = 0.4$ and $r_{23} = 0.9$. From the results in [1] we might expect loop 1 to travel clockwise around loop 2 before being ejected behind it, and loops 2 and 3 to exchange their amplitudes but never overlap. As can be seen from figure 3 this is indeed what happens. In this case $\delta_1 = 19.50$, $\delta_2 = -0.54$ and $\delta_3 = 3.18$.

Second, consider the case where $k_1 = 0.35$, $k_2 = 0.5$ and $k_3 = 1$ so that $r_{12} = 0.7$ and $r_{23} = 0.5$. From the results in [1] we might expect loops 1 and 2 to overlap for part of the interaction and to exchange their amplitudes during the interaction, and for loop 2 to travel clockwise around loop 3. As can be seen from figure 4 not only does this happen but after loops 1 and 2 overlap, they both travel around loop 3 before both are ejected behind loop 3. In this case $\delta_1 = 13.38$, $\delta_2 = 9.24$ and $\delta_3 = 0.10$.

Third, consider the case where $k_1 = 0.405$, $k_2 = 0.45$ and $k_3 = 1$ so that $r_{12} = 0.9$ and $r_{23} = 0.45$. From the results in [1] we might expect loops 1 and 2 to exchange their amplitudes

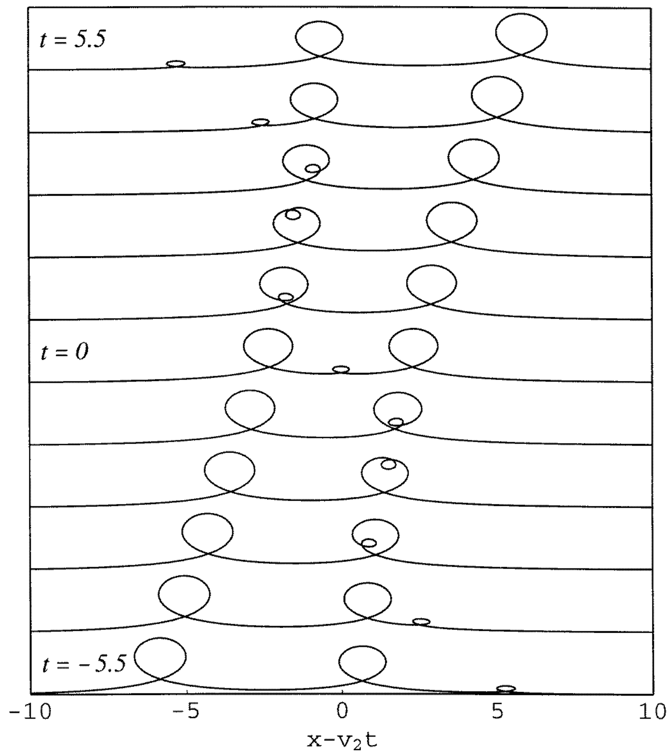

Figure 3. The interaction process for three loop solitons with $r_{12} = 0.4$ and $r_{23} = 0.9$.

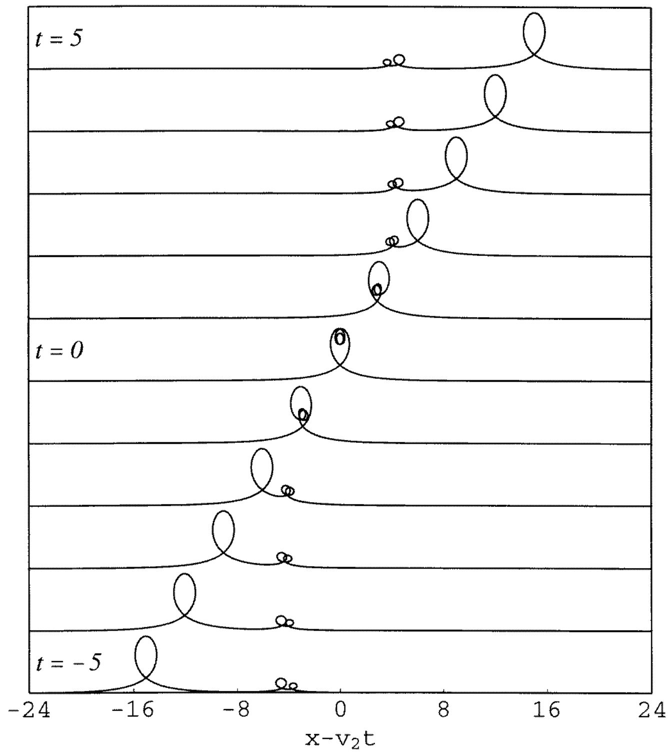

Figure 4. The interaction process for three loop solitons with $r_{12} = 0.7$ and $r_{23} = 0.5$.

but never overlap, and loop 2 to travel clockwise around loop 3. As can be seen from figure 5 both loops 1 and 2 travel clockwise around loop 3 and exchange amplitudes, but they overlap for a while near and at $t = 0$. In this case $\delta_1 = 9.77$, $\delta_2 = 10.97$ and $\delta_3 = 0.08$.

Clearly many other types of interaction are possible and, as demonstrated by our third example, it is not always possible to predict what will happen on the basis of the results in [1] alone. The interaction process for the three loop soliton solution is more complicated than that for the two loop soliton solution; we have been unable to classify the interactions into distinct characteristic cases for ranges of values of the ratios r_{12} and r_{23} in a way similar to that for the two loop solution. Nevertheless, the results from [1] can give us a rough indication as to what might happen during the interaction process for the three loop soliton solution.

6. Conclusion

We have found the N loop soliton solution to the VE by using a blend of transformations and Hirota's method. The one and two loop soliton solutions can be obtained indirectly by using certain elements of the inverse scattering transform (IST) method as applied to the KdV equation [9]. A direct application of the IST method to obtain the N loop soliton solution is currently under investigation. In this context it is of interest to note that the spectral problem associated with (2.3) is of third order [10] and may be obtained by using a Bäcklund transformation derived from (2.5) and (2.6).

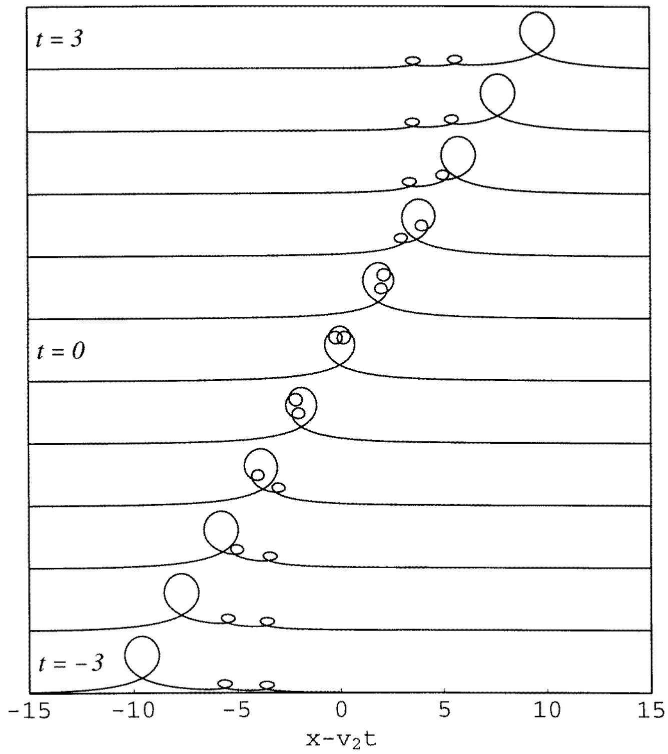

Figure 5. The interaction process for three loop solitons with $r_{12} = 0.9$ and $r_{23} = 0.45$.

Appendix. The N-soliton condition

For there to be an N-soliton solution (NSS) to (2.5) with $N(\geqslant 1)$ arbitrary, $F(D_X, D_T)$ must satisfy the 'N-soliton condition' (NSC) [2], namely

$$G^{(n)}(p_1, \ldots, p_n) = 0, \qquad n = 1, 2, \ldots, N, \tag{A.1}$$

where

$$G^{(1)}(p_1) := 0 \tag{A.2}$$

and, for $n \geqslant 2$,

$$G^{(n)}(p_1, \ldots, p_n) := C \sum_{\sigma=\pm 1} \left\{ F\left(\sum_{i=1}^{n} \sigma_i p_i, \sum_{i=1}^{n} \sigma_i \Omega_i \right) \times \prod_{i>j}^{(n)} F(\sigma_i p_i - \sigma_j p_j, \sigma_i \Omega_i - \sigma_j \Omega_j) \sigma_i \sigma_j \right\}. \tag{A.3}$$

In (A.3) the Ω_i are given in terms of the p_i by the dispersion relations $F(p_i, \Omega_i) = 0$ $(i = 1, \ldots, N)$, $\sum_{\sigma=\pm 1}$ means the summation over all possible combinations of $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1, \ldots, \sigma_n = \pm 1$, and C is a function of the p_i that is independent of the summation indices $\sigma_1, \ldots, \sigma_n$.

From (A.2) it follows that (A.1) is satisfied for $n = 1$. If $F(p, \Omega) = F(-p, -\Omega)$, which is true of (2.6), then (A.1) is satisfied for $n = 2$. Hence there is a 2SS. However, whether or not (A.1) is satisfied for $n \geqslant 3$ depends on the particular form of $F(p, \Omega)$.

With F given by (2.6), the dispersion relations give $\Omega_i = -1/p_i$ and (A.3) may be written

$$G^{(n)}(p_1,\ldots,p_n) := \left(\prod_{i=1}^{n} p_i\right) \sum_{\sigma=\pm1} \left\{ \left(\sum_{i=1}^{n} \sigma_i p_i\right)^2 \left[1 - \left(\sum_{i=1}^{n} \frac{\sigma_i}{p_i}\right)\left(\sum_{i=1}^{n} \sigma_i p_i\right)\right] \times \prod_{i>j}^{(n)} (\sigma_i p_i - \sigma_j p_j)^2 (p_i^2 + p_j^2 - \sigma_i\sigma_j p_i p_j) \right\}. \tag{A.4}$$

The presence of the first product term in (A.4) ensures that $G^{(n)}$ is a homogeneous polynomial in the p_i.

In passing we remark that previous work suggests, but does not prove, that (2.5) with (2.6) does have an NSS for all $N \geqslant 1$. The expression for F given by (2.6) is a special case of one proposed by Ito [4, equation (B.10)]. Ito claimed that this F satisfies the 3SC. Hietarinta [5] performed a search for bilinear equations of the form (2.5) that have an F that satisfies the 3SC. One such F was found to be the one given by (2.6). Hietarinta [6] later claimed that this F also passed the 4SC. The bilinear equation (2.5) with F given by (2.6) is a special case of one given in Grammaticos *et al* [7, equation (4.4)]; they showed that this equation has the Painlevé property. According to Hietarinta [6] a bilinear equation that has a 4SS and the Painlevé property is almost certainly integrable. All this evidence suggests that it is highly likely that (2.5) with (2.6) does have an NSS for all $N \geqslant 1$. Here we remove any doubt by using induction to prove that the condition (A.1) is satisfied with $G^{(n)}$ given by (A.4).

We need the following properties of $G^{(n)}$ (as given by (A.4)) for $n \geqslant 3$:

(i) $G^{(n)}(p_1,\ldots,p_n)|_{p_1=0} \equiv 0$,

(ii) $G^{(n)}(p_1,\ldots,p_n)|_{p_1=\pm p_2} = \pm 24 p_1^6 [\prod_{i=3}^{n} (p_i^2 - p_1^2)^2 (p_i^4 + p_1^4 + p_i^2 p_1^2)] G^{(n-2)}(p_3,\ldots,p_n)$

(iii) $G^{(n)}(p_1,\ldots,p_n)|_{p_1^2+p_2^2 \pm p_1p_2=0} = \pm(p_1 \mp p_2)(p_1^2 - p_2^2)(p_1^2 + p_2^2 \mp p_1p_2)$
$\times \left[\prod_{i=3}^{n} [\{(p_1 \pm p_2)^2 + p_i^2\}^2 - (p_1 \pm p_2)^2 p_i^2]\right] G^{(n-1)}(p_1 \pm p_2, p_3,\ldots,p_n)$.

(We established property (iii) by adapting the argument used to obtain equation (28) in [8] in the context of a shallow water wave equation.) Furthermore, because of the σ summation in (A.4), $G^{(n)}$ is an odd, symmetric function of the p_i. As already noted, the condition (A.1) is satisfied for $n = 1$ and $n = 2$. We now assume that the condition is satisfied for all $n \leqslant m - 1$, where $m \geqslant 3$; then the properties of $G^{(n)}$ imply that it may be factorized as follows:

$$G^{(m)}(p_1,\ldots,p_m) = \left[\prod_{i=1}^{m} p_i\right] \left[\prod_{i>j}^{(m)} (p_i^2 - p_j^2)^2 (p_i^2 + p_j^2 + p_i p_j)(p_i^2 + p_j^2 - p_i p_j)\right] \times \tilde{G}^{(m)}(p_1,\ldots,p_m), \tag{A.5}$$

where $\tilde{G}^{(m)}$ is a homogeneous polynomial. It follows that the degree of $G^{(m)}$ is at least $4m^2 - 3m$. On the other hand, from (A.4) the degree of $G^{(m)}$ is at most $2m^2 - m + 2$. As $4m^2 - 3m > 2m^2 - m + 2$ for $m \geqslant 3$, it follows that $G^{(m)} \equiv 0$. It now follows by induction that the NSC is satisfied.

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