Available online at www.sciencedirect.com

SCIENCE DIRECT®

Chaos, Solitons and Fractals 26 (2005) 1309–1316

CHAOS SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Explicit solutions of the Camassa–Holm equation

E.J. Parkes [a,*], V.O. Vakhnenko [b]

[a] Department of Mathematics, University of Strathclyde, Livingstone Tower, Richmond Street, Glasgow G1 1XH, UK
[b] Institute of Geophysics, Ukrainian Academy of Sciences, 03680 Kyiv, Ukraine

Accepted 29 March 2005

Communicated by Prof. M. Wadati

Abstract

Explicit travelling-wave solutions of the Camassa–Holm equation are sought. The solutions are characterized by two parameters. For propagation in the positive x-direction, both periodic and solitary smooth-hump, peakon, cuspon and inverted-cuspon waves are found. For propagation in the negative x-direction, there are solutions which are just the mirror image in the x-axis of the aforementioned solutions. Some composite wave solutions of the Degasperis–Procesi equation are given in an appendix.

1. Introduction

The family of equations

$$u_t - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{1.1}$$

where $b > 1$ is a constant, was discussed in [1]. Phase portraits were used to categorize travelling-wave solutions. In [2] the family was dubbed the 'peakon b-family'.

As discussed in [3], the family of Eq. (1.1) contains only two integrable equations, namely the dispersionless Camassa–Holm equation (CHE) for which $b = 2$ [4] and the Degasperis–Procesi equation (DPE) for which $b = 3$ [5].

It has been known for some time that the dispersionless Camassa–Holm equation, namely

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{1.2}$$

has a weak solution in the form of a single peakon [4]

$$u(x,t) = ve^{-|x-vt|}, \tag{1.3}$$

* Corresponding author. Tel.: +44 0 141 552 8657.
E-mail addresses: ejp@maths.strath.ac.uk (E.J. Parkes), vakhnenko@bitp.kiev.ua (V.O. Vakhnenko).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2005.03.011

where v is a constant, and an N-peakon solution [6] that is just a superposition of peakons, namely

$$u(x,t) = \sum_{j=1}^{N} p_j(t)\mathrm{e}^{-|x-q_j(t)|}, \tag{1.4}$$

where the $p_j(t)$ and $q_j(t)$ satisfy a certain associated dynamical system. (A substantial list of references regarding the properties of the CHE may be found in [7].)

Recently, a classification of travelling-wave solutions of the CHE was given in [7]. However, explicit solutions were given only for the solitary peakon and periodic peakon waves. Periodic smooth-hump waves and periodic cuspon waves were investigated numerically in [8].

The aim of the present paper is to present explicit solutions of the CHE for both periodic and solitary smooth-hump, smooth-well, peakon, inverted-peakon, cuspon and inverted-cuspon waves. We use a technique similar to the one we presented in [9] for the DPE.

In Section 2, we explain why the technique we used for the DPE in [9] also works for the CHE. In Section 3 we find explicit solutions for travelling waves and classify them in terms of two parameters. In Section 4 we give our concluding remarks.

2. An integrated form of Eq. (1.1)

In order to seek travelling-wave solutions to (1.1), it is convenient to introduce a new dependent variable z defined by

$$z = (u - v)/|v| \tag{2.1}$$

and to assume that z is an implicit or explicit function of η, where

$$\eta = x - vt - x_0, \tag{2.2}$$

v and x_0 are arbitrary constants, and $v \neq 0$. Then (1.1) becomes

$$zz_{\eta\eta\eta} + bz_\eta z_{\eta\eta} - (b+1)zz_\eta - bcz_\eta = 0, \quad \text{where } c := v/|v| = \pm 1. \tag{2.3}$$

After two integrations (2.3) gives

$$(zz_\eta)^2 = f(z), \tag{2.4}$$

where

$$f(z) := z^4 + 2cz^3 + Az^2 + Bz^{3-b}, \tag{2.5}$$

and A and B are real constants.

In [9] we showed that, when $f(z)$ is a quartic, (2.4) has two implicit solutions in which z and η are expressed in terms of a parameter w. For reference, these solutions are summarized in Appendix A.

Note that for $b > 1$, $f(z)$ is a quartic for $b = 2$ or $b = 3$ only. The case $b = 3$, for which (1.1) is the DPE, was considered in [9]. Explicit periodic and solitary-wave solutions were found. Because $f(z)$ is also a quartic when $b = 2$, we can use a similar technique in order to investigate the CHE.

3. Explicit travelling-wave solutions of the CHE

When $b = 2$, (2.5) becomes

$$f(z) := z^4 + 2cz^3 + Az^2 + Bz \equiv (z - z_1)(z - z_2)(z_3 - z)(z_4 - z). \tag{3.1}$$

For the solutions of (2.4) that we are seeking, z_1, z_2, z_3 and z_4 are real constants with $z_1 \leqslant z_2 \leqslant z \leqslant z_3 \leqslant z_4$ and $z_1 + z_2 + z_3 + z_4 = -2c$.

From (3.1) it can be seen that one of z_1, z_2, z_3 and z_4 is always zero. We let the other three be q, r and s, where $s \leqslant r \leqslant q$ and $s = -q - r - 2c$. The types of solution to (2.4) may be categorized by an appropriate choice of the two parameters q and r. In [7,8] the two parameters that were used, namely M and m in the notation of [7,8], are equivalent to $q + c$ and $r + c$ respectively.

Note that (2.4) is invariant under the transformation $z \to -z$, $c \to -c$; this corresponds to the transformation $u \to -u$, $v \to -v$ in (2.1). Here we will seek the family of solutions of (2.4) for which $v > 0$ in (2.2) and so, from here on in this section, we will assume that $c = 1$.

3.1. $z_4 = 0$: Periodic smooth hump with $v > 0$

Suppose $z_4 = 0$ so that $z_1 = s$, $z_2 = r$ and $z_3 = q$. Consider the case $z_1 < z_2 < z_3 < 0$ so that

$$-q - r - 2c < r < q < 0. \tag{3.2}$$

(This is equivalent to the case considered numerically in Section 4.1 of [8].) The solution to (2.4) is a periodic hump given by (A.1) or (A.4) with $r \leqslant z \leqslant q$ and $0 < m < 1$; see Fig. 1 for an example given by (A.1).

3.2. $z_4 = 0$: Solitary smooth hump with $v > 0$

In Section 3.1 consider the limit $z_1 = z_2$ so that we have $m = 1$ and $z_1 = z_2 < z_3 < 0$. In this case

$$-c < r < -\frac{2}{3}c, \quad q = -2(r + c). \tag{3.3}$$

The solution to (2.4) is a smooth-hump solitary wave given by (A.6) with $r < z \leqslant -2(r + c)$; see Fig. 2 for an example.

3.3. $z_4 = 0$: Periodic peakon with $v > 0$

In Section 3.1 consider the limit $z_3 = z_4$ so that we have $m = 1$ and $z_1 < z_2 < z_3 = 0$. In this case

$$-c < r < 0, \quad q = 0. \tag{3.4}$$

The solution to (2.4) is given by (A.3) and has $r \leqslant z \leqslant 0$. From this we can construct a weak solution, namely the periodic peakon wave given by

$$z = z(\eta - 2j\eta_m), \quad (2j - 1)\eta_m \leqslant \eta \leqslant (2j + 1)\eta_m, \quad j = 0, \pm 1, \pm 2, \ldots \tag{3.5}$$

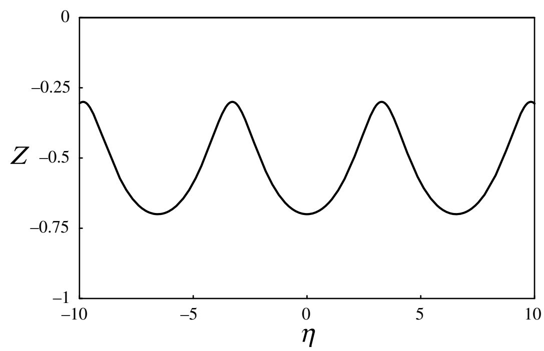

Fig. 1. Periodic smooth hump with $r = -0.7$, $q = -0.3$ and $v > 0$.

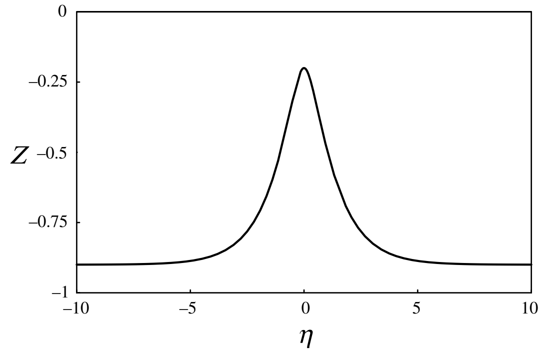

Fig. 2. Solitary smooth hump with $r = -0.9$, $q = -0.2$ and $v > 0$.

where

$$z(\eta) := [z_2 - z_1 \tanh^2(\eta/2)]\cosh^2(\eta/2) = -c + (r + c)\cosh\eta \tag{3.6}$$

and

$$\eta_m := 2\tanh^{-1}\left(\sqrt{\frac{z_2}{z_1}}\right) = 2\tanh^{-1}\left(\sqrt{\frac{-r}{r+2c}}\right); \tag{3.7}$$

see Fig. 3 for an example. The solution given by (3.5)–(3.7) is the spatially periodic solution of the CHE that has been dubbed a 'coshoidal wave' by Boyd [10].

3.4. $z_4 = 0$: Solitary peakon with $v > 0$

In Section 3.1 consider the limit $z_1 = z_2$ and $z_3 = z_4$ so that we have $z_1 = z_2 < z_3 = 0$ and then

$$r = -c, \quad q = 0. \tag{3.8}$$

In this case neither (A.3) nor (A.6) is appropriate. Instead we consider (2.4) with $f(z) = z^2(z + c)^2$ and note that the bound solution has $-c < z \leqslant 0$. On integrating (2.4) and setting $z = 0$ at $\eta = 0$ we obtain the weak solution

$$z = c(e^{-|\eta|} - 1), \tag{3.9}$$

i.e. a solitary peakon with amplitude c; see Fig. 4.

3.5. $z_3 = 0$: Periodic cuspon with $v > 0$

Suppose $z_3 = 0$ so that $z_1 = s$, $z_2 = r$ and $z_4 = q$. First let us consider the case $z_1 < z_2 < 0 < z_4$ so that

$$-q - r - 2c < r < 0 < q. \tag{3.10}$$

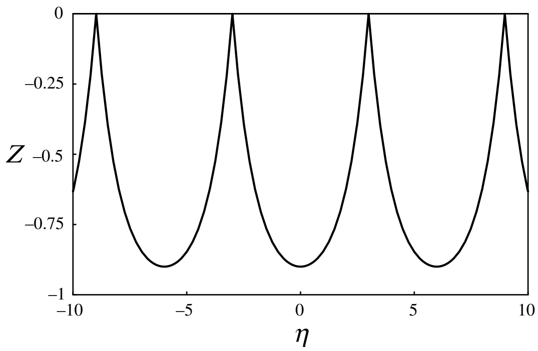

Fig. 3. Periodic peakon with $r = -0.9$, $q = 0$ and $v > 0$.

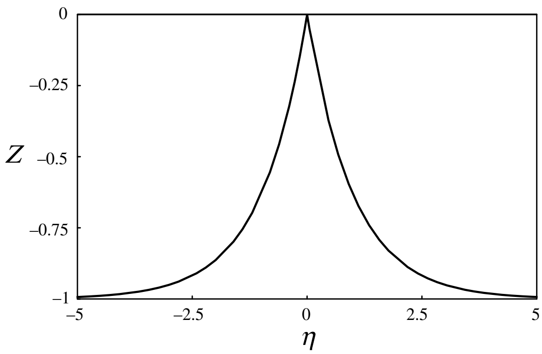

Fig. 4. Solitary peakon with $r = -1$, $q = 0$ and $v > 0$.

(This is equivalent to the case considered numerically in Section 4.2 of [8].) The solution to (2.4) is a periodic cuspon given by (A.1) or (A.4) with $r \leqslant z \leqslant 0$ and $0 < m < 1$; see Fig. 5 for an example given by (A.4).

3.6. $z_3 = 0$: Solitary cuspon with $v > 0$

In Section 3.5 consider the limit $z_1 = z_2$ so that we have $m = 1$ and $z_1 = z_2 < 0 < z_4$. In this case

$$r < -c, \quad q = -2(r + c). \tag{3.11}$$

The solution to (2.4) is a solitary cuspon given by (A.6) with $r < z \leqslant 0$; see Fig. 6 for an example.

3.7. $z_2 = 0$: Periodic inverted cuspon with $v > 0$

Suppose $z_2 = 0$ so that $z_1 = s$, $z_3 = r$ and $z_4 = q$. First let us consider the case $z_1 < 0 < z_3 < z_4$ so that

$$-q - r - 2c < 0 < r < q. \tag{3.12}$$

The solution to (2.4) is a periodic inverted cuspon given by (A.1) or (A.4) with $0 \leqslant z \leqslant r$ and $0 < m < 1$; see Fig. 7 for an example given by (A.1).

3.8. $z_2 = 0$: Solitary inverted cuspon with $v > 0$

In Section 3.7 consider the limit $z_3 = z_4$ so that we have $m = 1$ and $z_1 < 0 < z_3 = z_4$. In this case

$$0 < r = q. \tag{3.13}$$

The solution to (2.4) is a solitary inverted cuspon given by (A.3) with $0 \leqslant z < r$; see Fig. 8 for an example.

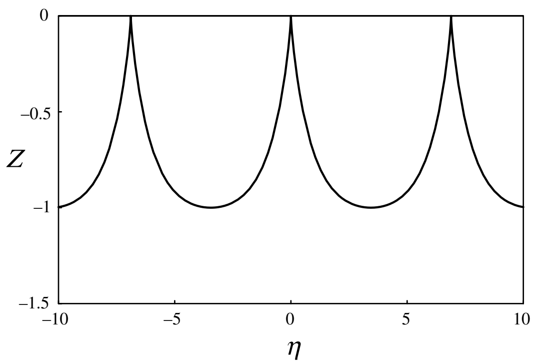

Fig. 5. Periodic cuspon with $r = -1$, $q = 0.1$ and $v > 0$.

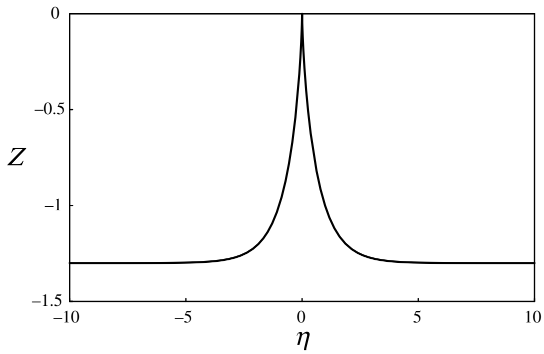

Fig. 6. Solitary cuspon with $r = -1.3$, $q = 0.6$ and $v > 0$.

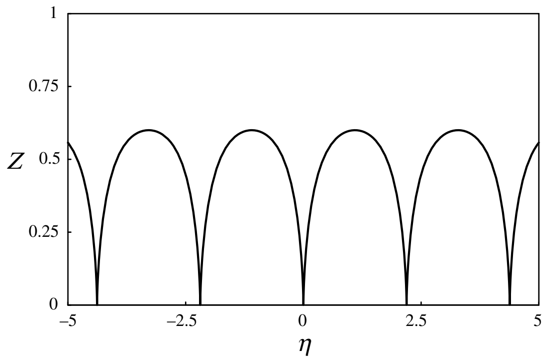

Fig. 7. Periodic inverted cuspon with $r = 0.6$, $q = 0.7$ and $v > 0$.

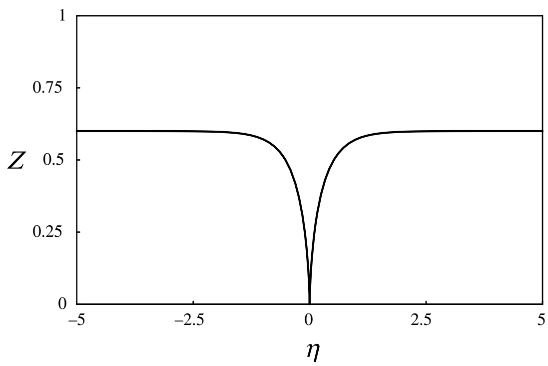

Fig. 8. Solitary inverted cuspon with $r = q = 0.6$ and $v > 0$.

3.9. $z_1 = 0$ and $v > 0$

In this case $z_2 + z_3 + z_4 > 0$ and so the condition $z_2 + z_3 + z_4 = -2c$ cannot be satisfied. Hence there are no solutions with $z_1 = 0$.

4. Concluding remarks

In Sections 3.1–3.8 we have found explicit expressions for eight different travelling-wave solutions to the CHE that travel in the positive x-direction with speed v, i.e. with $v > 0$. These solutions depend on two parameters q and r. For each of the aforementioned solutions expressed with u as the dependent variable, there is a solution for u that is the mirror image in the x-axis and travels with the same speed but in the opposite direction, i.e. with $v < 0$. For example, the mirror image of categories 3.2 and 3.5 respectively are solitary smooth wells with $v < 0$ and periodic inverted cuspons with $v < 0$.

In Theorem 1 in [7], Lenells categorized travelling-wave solutions to the CHE. His categories (a)–(d) correspond to our categories 3.1–3.4. His category (e), i.e. periodic cuspons, correspond to our category 3.5, i.e. periodic cuspons with $v > 0$, together with the mirror image of our category 3.7, i.e. periodic cuspons with $v < 0$. His category (f), i.e. solitary cuspons, correspond to our category 3.6, i.e. solitary cuspons with $v > 0$, together with the mirror image of our category 3.8, i.e. solitary cuspons with $v < 0$. His categories (a′)–(f′) are the mirror images of his categories (a)–(f) respectively.

In [9] we investigated the DPE. As for the CHE, for $v > 0$ we found explicit expressions for smooth-hump and peakon solitary waves and their periodic equivalents. Unlike the CHE for which we have found cuspon and inverted-cuspon solutions, we showed in [9] that the DPE has inverted loop-like solutions instead. However, it should be noted that it is possible to construct other explicit solutions for the DPE as composite waves by using the results in [9]. These are summarized in Appendix B.

Appendix A

As shown in [9], there are two solutions to (2.4) when $f(z)$ is a quartic. These are summarized below.

The first solution is

$$z = \frac{z_2 - z_1 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)}, \qquad \eta = \frac{1}{p}[wz_1 + (z_2 - z_1)\Pi(n; w|m)], \tag{A.1}$$

where

$$m = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)}, \quad n = \frac{z_3 - z_2}{z_3 - z_1}, \quad p = \frac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)}. \tag{A.2}$$

In (A.1) $\mathrm{sn}(w|m)$ is a Jacobian elliptic function and the notation is as used in [11, Chapter 16]; $\Pi(n;w|m)$ is the elliptic integral of the third kind and the notation is as used in [11, Section 17.2.15]. When $z_3 = z_4$, $m = 1$ and so (A.1) becomes

$$z = \frac{z_2 - z_1 n\tanh^2 w}{1 - n\tanh^2 w}, \qquad \eta = \frac{wz_3}{p} - 2\tanh^{-1}(\sqrt{n}\tanh w). \tag{A.3}$$

The second solution is

$$z = \frac{z_3 - z_4 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)}, \qquad \eta = \frac{1}{p}[wz_4 - (z_4 - z_3)\Pi(n; w|m)], \tag{A.4}$$

where m and p are as in (A.2) but n is given by

$$n = \frac{z_3 - z_2}{z_4 - z_2}. \tag{A.5}$$

When $z_1 = z_2$, $m = 1$ and so (A.4) becomes

$$z = \frac{z_3 - z_4 n\tanh^2 w}{1 - n\tanh^2 w}, \qquad \eta = \frac{wz_2}{p} + 2\tanh^{-1}(\sqrt{n}\tanh w). \tag{A.6}$$

The solutions (A.1) and (A.4) have the same profile but the former has $z = z_2$ when $\eta = 0$ whereas the latter has $z = z_3$. (A.4) may be obtained from (A.1) by shifting w by $K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. As indicated in obtaining (A.3) and (A.6) above, (A.1) is the appropriate solution to use when considering the limit $z_3 = z_4$ whereas (A.4) is appropriate when considering the limit $z_1 = z_2$.

Appendix B

As a footnote to the analysis of the DPE in [9], we give some explicit composite wave solutions for which $v > 0$. The notation here refers to that in [9].

For $A < 0$ and $B = B_U$, there is a solitary loop-like wave as shown in Fig. 2(a). If the part of the loop for which $z < 0$ is removed, the remaining parts may be joined to form a solitary inverted cuspon with $0 \leqslant z < z_U$. For $A < 0$ and $0 < B < B_U$, a similar procedure applied to the periodic inverted loop-like waves shown in Fig. 1(a) leads to periodic inverted cuspons with $0 \leqslant z \leqslant z_3$.

If in Fig. 2(a) the part of the loop for which $z > 0$ is removed, the remaining part may be repeated periodically to give a periodic cuspon wave with $z_2 \leqslant z \leqslant 0$. If in Fig. 1(a) the part of the loops for which $z > 0$ is removed, the remaining parts may be joined to give a periodic cuspon wave with $z_2 \leqslant z \leqslant 0$.

The waves described above have explicit expressions that may be obtained from the ones given in [9]. However, we note that the DPE has other solutions for which we have not found explicit expressions: for $A < 1$ and $B = B_L$ there is a solitary cuspon with $z_L < z \leqslant 0$; for $A < 0$ and $B_U < B < B_L$ there is a family of periodic cuspons with $z_2 \leqslant z \leqslant 0$; for $0 \leqslant A < 1$ and $0 < B < B_L$ there is a family of periodic cuspons with $z_2 \leqslant z \leqslant 0$.

References

[1] Guo B, Liu Z. Periodic cusp wave solutions and single-solitons for the *b*-equation. Chaos, Solitons & Fractals 2004;23:1451–63.

[2] Holm DD, Hone ANW. A class of equations with peakon and pulson solutions (with an appendix by Harry Braden and John Byatt-Smith). J Nonlinear Math Phys 2005;12(Suppl. 1):380–94.

[3] Degasperis A, Holm DD, Hone ANW. A new integrable equation with peakon solutions. Theor Math Phys 2002;133:1463–74.

[4] Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. Phys Rev Lett 1993;71:1661–4.

[5] Degasperis A, Procesi M. Asymptotic integrability. In: Degasperis A, Gaeta G, editors. Symmetry and perturbation theory. Singapore: World Scientific; 1999. p. 23–37.

[6] Camassa R, Holm DD, Hyman JM. A new integrable shallow water equation. Adv Appl Mech 1994;31:1–33.

[7] Lenells J. Traveling wave solutions of the Camassa–Holm equation. J Diff Eq (in press).

[8] Kalisch H, Lenells J. Numerical study of traveling-wave solutions for the Camassa–Holm equation. Chaos, Solitons & Fractals 2005;25:287–98.

[9] Vakhnenko VO, Parkes EJ. Periodic and solitary-wave solutions of the Degasperis–Procesi equation. Chaos, Solitons & Fractals 2004;20:1059–73.

[10] Boyd JP. Peakons and coshoidal waves: traveling wave solutions of the Camassa–Holm equation. Appl Math Comput 1997;81:173–87.

[11] Abramowitz M, Stegun IA. Handbook of mathematical functions. New York: Dover; 1972.