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Increase of Nonlinear Effect in a Medium with Microstructure

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Abstract—An averaged set of hydrodynamic equations in the Lagrangian and Eulerian coordinates is obtained for the description of long nonlinear waves in a structured medium. The nature of the long-wave effect is different at different scale levels. At the microscopic level, the behavior of the medium is governed by the laws of thermodynamics, while at the macroscopic level it manifests itself as a wave motion for the average characteristics. An evolutionary equation allowing for weak structure-caused nonlinearity is obtained. It is shown that the medium structure gives rise to an increase in the nonlinear effects that accompany the propagation of long waves.

INTRODUCTION

Many natural media have natural internal structure. Recent experimental studies have demonstrated that internal structure of a medium affects wave motion [1–7]. The presence of inhomogeneities makes the problem more complex, and, at the same time, it manifests itself in a more general form in the propagation of nonlinear waves. For example, such phenomena as (a) soliton-like properties of the *P*-wave [8] and (b) the increase of nonlinear effects in structured media as compared to homogeneous ones [3–7] can be considered as nonlinear manifestations of wave processes in natural media.

Models of various degree of complexity are used to describe the wave processes in heterogeneous media. It is a common idea that perturbations with the wavelength λ significantly exceeding the size of the structural inhomogeneities ε' propagate in the medium as in a homogeneous one. The known idealization of a real medium with the help of a homogeneous one within the framework of mechanics of continuum was quite successful in the case of the description of wave processes (e.g., [9–11]). In terms of acoustics, it is possible to take into account the structure of a medium within the framework of models of homogeneous media with certain dispersion–dissipation properties [12, 13]. Continuum models are also used for the description of nonlinear waves [14–17]. At this level, media are simulated within the framework of elastic, viscoelastic, and elastoplastic homogeneous media [15, 18]. In these cases, the medium structure is taken into account indirectly through kinetic parameters (relaxation time, viscosity coefficients, etc.) [5, 6, 9, 14–18].

A model of interpenetrating continua developed for the description of dynamic behavior of multicomponent media is substantiated using the methods of classical mechanics of continua [19] and statistical physics [20]. The fundamental assumption in the theory of mixtures [14] coincides with the assumption underlying the model of interpenetrating continua [19]. Namely, it is assumed that each microscopic volume dv contains particles of each component. Equations of motion written for each component include the terms describing the mass, force, and thermal interactions between components. The problem becomes more complex because, in the general case, it is necessary to use experimental data in order to establish theoretical relationships between microscopic parameters at the level of component interaction. A review and methods of application of various models to the description of wave processes in mixtures are presented in the book by Rajagopal and Tao [14].

All mentioned models use the formalism of mechanics of continuous media. They are based on the principle of local action and the generalization of mechanics' laws valid for a mass point to a continuous medium [10].

When we change from integral conservation equations to differential equations, we rely on the existence of a differentially small microscopic volume dv. On the one hand, this volume is so small that laws of mechanics of a point mass can be extended to the whole microscopic volume. But on the other hand, though a microscopic volume is small in comparison with the whole volume occupied by the medium, it contains a great number of structural elements of the medium, and in this sense it may be considered as a macroscopic one. Thus, the transition to differential conservation equations is based on the assumption of smallness of the microstructure scale ε' in comparison with the characteristic macroscopic scale of flow λ. In this case, the passage to the limit $\varepsilon'/\lambda \longrightarrow 0$ must be performed. A compression of the volume dv to a point is correct in the general case for continuous functions [10, 14]. This means that all points within a differentially small vol-

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ume are equivalent. Therefore, the equivalence of points within a microscopic volume implies that wave field characteristics averaged over dv must be used. Hence, it is assumed that equations of motion can be written for average characteristics, such as density, mass velocity, and pressure, which are assigned to each component separately. It is necessary to note that the size of components is not included in these models explicitly.

Application of models of continuous media to the description of dynamic wave processes in a structured medium has certain fundamental difficulties [1, 2, 6, 18]. In this paper, the medium structure is taken into account at the macroscopic level. We reject the assumption that a differentially small volume dv contains all medium components, though we consider long-wave approximations, when the wavelength λ far exceeds the characteristic length of the medium structure ε'. As it is assumed, a single component of a structured medium is simulated by a homogeneous medium (a differentially small volume dv is much smaller than the characteristic size of a single component). A rigorous mathematical analysis by the method of asymptotic averaging [21–23] shows that the medium structure directly affects the nonlinear wave processes even in the case of perturbations with the wavelength much greater than the size of inhomogeneities. The mathematical formulation of this statement means that the set of averaged equations is not expressed in averaged characteristics (pressure, mass velocity, and specific volume) and contains the terms with the characteristic size of single components.

This study is devoted to the analysis of weak nonlinearity related to the medium structure in the case of propagation of long-wave perturbations. An asymptotic averaged model [24– 30] is used to describe the wave processes in media with microstructure. It is shown that long-wave propagation can be described only at the acoustic level, with the help of dispersion–dissipation properties of a medium, and only in this case the dynamic behavior of a medium can be simulated in terms of homogeneous relaxation [26, 30]. At the same time, a long wave of finite amplitude responds to the structure of a medium in such way that the behavior of structured medium cannot be simulated by a homogeneous medium. An important result predicted by this model is an increase of nonlinear effect in the propagation of a wave of finite amplitude in a medium with microstructure, even if single components of the medium are described by a linear law.

A SET OF AVERAGED EQUATIONS

Elementary inhomogeneous media which give an opportunity to analyze the structure effect are media with a regular structure. Propagation laws of long-wave perturbations are studied for a periodic medium under the condition of equality of both pressure and mass velocity at the boundary of neighboring components. It is assumed that an element of the medium microstructure ε is large enough to allow the application of the laws of classical mechanics of continuous media. The medium is barotropic. We assume that periodically variable properties are such properties of an unperturbed medium as the specific volume $V = \rho^{-1}$ and the sound velocity c (although this assumption turns out to be unimportant for the final result in the case of long waves). We use a hydrodynamic approach and consider media without tangential stress. This restriction is justified in the case of simulation of nonlinear waves in water-saturated soil, bubble media, aerosols, etc. The consideration may be extended to solids in the case of investigation of powerful loads under the conditions when it is possible to ignore the strength and plastic properties of material [31].

In the Lagrangian coordinates (m is the mass spatial coordinate), equations of plane one-dimensional motion for each element of a medium with regular structure have the form

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial m} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial m} = 0. \tag{1}$$

Common notations are used here. The conditions of matching are the equality of mass velocities and the equality of pressures at the component boundaries. Equations of state are known for each component:

$$dp = c^2 d\rho. \tag{2}$$

One of the ways to study such an inhomogeneous medium is the method of asymptotic averaging of equations with fast-oscillating coefficients [21–23]. The main idea of the asymptotic method consists in the application of the method of multiple scales in combination with the method of spatial averaging. The small parameter of the problem is $\varepsilon = \varepsilon'/\lambda$. According to this method, the mass spatial coordinate m is divided into two independent coordinates: the slow coordinate s and the fast coordinate ξ. In this case

$$m = s + \varepsilon\xi, \quad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{3}$$

The slow coordinate s corresponds to the global change of wave field and is constant within the whole period, while the fast variable ξ follows the field variations within the period of the structure. The dependent functions p, u, and V are represented in the form of power series expansions in the structure period ξ, for example,

$$p(m, t) = p^{(0)}(s, t, \xi) + \varepsilon p^{(1)}(s, t, \xi) + \varepsilon^2 p^{(2)}(s, t, \xi) + \dots. \tag{4}$$

The functions $p^{(i)}$, $u^{(i)}$, and $V^{(i)}$ are considered as single-periodic in ξ.

Let us prove that $p^{(0)} = p^{(0)}(s, t)$, $p^{(1)} = p^{(1)}(s, t)$, and $u^{(0)} = u^{(0)}(s, t)$ do not depend on the fast variable ξ. After substitution of expressions (3) and (4) in the initial equations of motion (1), we have

$$-\varepsilon^{-1}\frac{\partial u^{(0)}}{\partial \xi}+\varepsilon^{0}\left(\frac{\partial V^{(0)}}{\partial t}-\frac{\partial u^{(0)}}{\partial s}-\frac{\partial u^{(1)}}{\partial \xi}\right)$$
$$+\varepsilon^{1}\left(\frac{\partial V^{(1)}}{\partial t}-\frac{\partial u^{(1)}}{\partial s}-\frac{\partial u^{(2)}}{\partial \xi}\right)+\ldots = 0,$$
$$\varepsilon^{-1}\frac{\partial p^{(0)}}{\partial \xi}+\varepsilon^{0}\left(\frac{\partial u^{(0)}}{\partial t}+\frac{\partial p^{(0)}}{\partial s}+\frac{\partial p^{(1)}}{\partial \xi}\right)$$
$$+\varepsilon^{1}\left(\frac{\partial u^{(1)}}{\partial t}+\frac{\partial p^{(1)}}{\partial s}+\frac{\partial p^{(2)}}{\partial \xi}\right)+\ldots = 0.$$

According to the general theory of the asymptotic method, the coefficients at different powers of ε must be equal to zero independently of each other. Thus, in the order $O(\varepsilon^{-1})$ we have $\frac{\partial p^{(0)}}{\partial \xi} = 0$ and $\frac{\partial u^{(0)}}{\partial \xi} = 0$, i.e., $p^{(0)} = p^{(0)}(s, t)$ and $u^{(0)} = u^{(0)}(s, t)$ do not depend on ξ. These properties can be expressed in the form $\langle u^{(0)}\rangle = u^{(0)}$ and $\langle p^{(0)}\rangle = p^{(0)}$. The following expressions must be valid in the order $O(\varepsilon^{0})$:

$$\frac{\partial V^{(0)}}{\partial t}-\frac{\partial u^{(0)}}{\partial s}-\frac{\partial u^{(1)}}{\partial \xi} = 0,$$
$$\frac{\partial u^{(0)}}{\partial t}+\frac{\partial p^{(0)}}{\partial s}+\frac{\partial p^{(1)}}{\partial \xi} = 0. \tag{5}$$

Now, we apply the procedure of averaging, which is possible only in the Lagrangian mass coordinates, since the period in this case does not depend on wave motion. We have $\langle\cdot\rangle = \int_0^1(\cdot)d\xi$ by definition. Here we use the normalization condition $\int_0^1 d\xi = 1$. Since $p^{(1)}$ and $u^{(1)}$ are periodic, we obtained $\left\langle\frac{\partial p^{(1)}}{\partial \xi}\right\rangle = 0$ and $\left\langle\frac{\partial u^{(1)}}{\partial \xi}\right\rangle = 0$. After the integration of equations (2) and (5) with respect to the structure period ξ, we have, on the one hand, an averaged set of equations [24–30]

$$\frac{\partial\langle V^{(0)}\rangle}{\partial t}-\frac{\partial u^{(0)}}{\partial s} = 0, \quad \frac{\partial u^{(0)}}{\partial t}+\frac{\partial p^{(0)}}{\partial s} = 0, \tag{6}$$

$$d\langle V^{(0)}\rangle = -\left\langle\frac{(V^{(0)})^{2}}{c^{2}}\right\rangle dp, \tag{7}$$

and, on the other hand, we obtain $\frac{\partial p^{(1)}}{\partial \xi} = 0$ as the result of subtraction of the second equation (6) from the second equation of set (5). This means that $p^{(1)}$ does not depend on ξ as well. In contrast to the quantities $u^{(0)}$, $p^{(0)}$, and $p^{(1)}$, the quantity $V^{(0)}$ is a function of ξ. Below, we restrict ourselves to the zero-order approximation and omit the upper index 0.

The averaged set of equations (6) and (7) is an integro-differential set, and, in the general case, it cannot be reduced to the averaged characteristics p, u, and $\langle V\rangle$.

Equations (6) and (7) are derived for a strictly periodic medium. However, it is possible to demonstrate that equations (6) and (7) are also valid for media with a quasi-periodic structure. The pressure p and the mass velocity u are constant within the whole structure period. A structural element moves as a whole (u = const). At the scale ξ, the load is so slow that there is enough time for the pressure balance to be established in the microscopic volume, the internal structure changes, and the mechanical equilibrium takes place (p = const). Therefore, the effect is homogeneous (waveless) over the entire period of the medium structure. At this hierarchical level, the medium behavior is governed only by thermodynamic laws.

At the macroscopic level, the medium behavior is described by the laws of wave dynamics (6) for the average characteristics u, p, and $\langle V\rangle$. However, the medium structure affects the wave motion, and this is caused by the following mechanisms. According to equation (7), the variation in the average specific volume $\langle V\rangle$ does not correspond to the variation in the specific volume for a homogeneous medium in the process of loading. Thus, a change in the internal structure causes some variation in the average specific volume $\langle V\rangle$, and finally, the medium structure manifests itself at the macroscopic level s as a wave motion despite the fact that equations of motion (6) are written for the average quantities u, p, and $\langle V\rangle$.

From the mathematical point of view, in the zero-order approximation in ε, the period length is infinitely small ($\varepsilon \longrightarrow 0$). This means that the positions of single components within the period are insignificant. The set of equations (6) and (7) does not change, if the positions of layers in the unit cell are changed, or the layers are broken into smaller parts. Therefore, equations (6) and (7) describe the behavior of any quasi-periodic (statistically inhomogeneous) medium, which has the same mass contents of components at the level of microstructure independently of the position of the substance in the cell volume.

This asymptotic averaged model justifies the one-velocity continuum models. In particular, the well-known model developed by Lyakhov for multicomponent media [17] received its rigorous mathematical justification [27, 29, 30].

A technique for the numerical solution of the set of equations is described in our previous papers [24–26]. In this technique, the step of integration is limited by the length of wave perturbation rather than the structure period. The main initial problem of computation is related to the smallness of the step of integration. This problem was solved successfully. Thus, the set of averaged equations can be solved at large distances within reasonable processor time. We developed a package of computer codes for solving this set of equations.

A SET OF EQUATIONS IN THE EULERIAN COORDINATES

It is convenient for many problems to represent equations (6) in the Eulerian coordinates. It is possible to express the averaged equations of motion (6) in terms of these independent variables by virtue of the fact that they involve the average characteristics u, p, and $\langle V \rangle$. We note that a direct application of the asymptotic method of averaging to equations in the Eulerian coordinates is improper, since the size of microstructures is not constant.

Let us determine transformations between the independent variables in the Eulerian coordinates (x, t_E) and the Lagrangian coordinates (s, t) [26, 30]

$$x = x(x, t), \quad t_E = t. \tag{8}$$

It is important that the velocity u does not change within the period, and therefore, we can deal with an average trajectory of a particle. The averaged Eulerian coordinate x for an individual particle (its trajectory) changes with time

$$\left(\frac{\partial x}{\partial t}\right)_s = u(s, t). \tag{9}$$

This relationship is the definition of slow Eulerian coordinate x. Moreover, x changes with varying s, i.e., it is necessary to represent transformation (8) in the differential form:

$$dx = Ads + udt, \quad t_E = t. \tag{10}$$

From physical considerations, it is clear that the position of a particle is determined unambiguously by the particle itself and time. Mathematically, this means that dx in expression (10) is an exact differential. Therefore, we obtain the condition

$$\frac{\partial A}{\partial t} = \frac{\partial u}{\partial s}.$$

This condition is satisfied at $A = \langle V \rangle$, because it passes into continuity equation (6).

Thus, we obtained a transformation between the Lagrangian and Eulerian coordinates

$$dx = \langle V \rangle ds + udt, \quad t_E = t. \tag{11}$$

In this case, partial derivatives change according to formulas

$$\frac{\partial}{\partial s} = \langle V \rangle \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t_E} + u\frac{\partial}{\partial x}.$$

Equations of motion (6) in the Eulerian coordinates take the form (the index E is omitted)

$$\frac{\partial \langle V \rangle^{-1}}{\partial t} + \frac{\partial u \langle V \rangle^{-1}}{dx} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \langle V \rangle \frac{\partial p}{\partial x} = 0. \tag{12}$$

It is convenient to determine the fast Eulerian coordinate ζ as

$$\left(\frac{\partial \zeta}{\partial \xi}\right)_t = \frac{\tilde{\rho}}{\rho(\xi)}. \tag{13}$$

We note that the averaged density $\tilde{\rho}$ in the Eulerian coordinates is the quantity commonly used for density [26],

$$\langle V \rangle = \int_0^1 V(\xi) d\xi = \int_0^1 V \frac{\rho(\xi)}{\tilde{\rho}} d\zeta = \tilde{\rho}^{-1}.$$

It is this quantity that is determined experimentally. At the same time, the average values of p and u in both coordinate systems coincide.

The averaged equations of motion in the Lagrangian as well as Eulerian coordinates are analogous in their form to the equations for a homogeneous medium in the corresponding coordinates. Only equations of state (2) and (7) are noticeably different. The medium structure is represented by equation (7). As it will be demonstrated below, the term $\langle V^2/c^2 \rangle$ introduces additional nonlinearity.

NONLINEAR EFFECTS IN A MEDIUM WITH MICROSTRUCTURE

Introduction of the effective average sound velocity according to the formula

$$c_{\text{eff}} = \sqrt{\frac{\langle V \rangle^2}{\langle V^2/c^2 \rangle}} \tag{14}$$

leads us to a conventional form of the set of equations. However, c_{eff} is not an averaged characteristic, i.e., $c_{\text{eff}}^2 \neq \langle c^2 \rangle$. It is evident, that the medium structure makes a certain contribution to nonlinearity. Even if sound velocity in each component is independent of pressure ($c \neq f(p)$), the quantity c_{eff} is a function of pressure in the general case.

At the same time, at the acoustic level the set of equations is reduced to the averaged characteristics u, p, and $\langle V \rangle$, because $\langle V^2/c^2 \rangle_0 \neq f(p)$ and the fields of pressure and mass velocity coincide in the periodic and homogeneous media if the properties are matched by the conditions [24–26]

$$\langle V \rangle_0 = V_0, \quad \langle V^2/c^2 \rangle_0 = \langle V_0^2/c_0^2 \rangle. \tag{15}$$

In our previous paper [30], we demonstrated that acoustic waves propagate in the same way as in a homogeneous medium, even if relaxation processes take place in the medium components.

Normalization to the average specific volume $\langle V \rangle_0$ and the initial effective sound velocity c_{eff} provides an opportunity to compare the results obtained for various media.

In contrast to acoustic waves, nonlinear waves with even longer wavelengths respond to the internal structure of the medium. It will be shown that nonlinear effects in these media become stronger in comparison with those in homogeneous media. Now, we consider an evolutionary equation with weak nonlinearity and compare the nonlinear coefficients of these media.

We obtain the evolutionary equation in the Eulerian coordinates with allowance for a weak nonlinearity. First of all, we note that the mass velocity u is related to the pressure p by the formula [25]

$$u = \int_{p_0}^{p} \sqrt{\langle V^2/c^2\rangle}\,dp. \quad (16)$$

The functional dependence of the average specific volume on the pressure increment $p' = p - p_0$ in the second order of magnitude, $O(p'^2)$, are presented in the form of a series

$$\langle V\rangle(p) = \langle V\rangle_0 + \left.\frac{d\langle V\rangle}{dp}\right|_{p=p_0} p' + \frac{1}{2}\left.\frac{d^2\langle V\rangle}{dp^2}\right|_{p=p_0} p'^2.$$

Then, the set of equations (7) and (12) can be represented in the form

$$\langle V\rangle_0\frac{\partial u}{\partial x} + \left\langle\frac{V^2}{c^2}\right\rangle_0\frac{\partial p'}{\partial t} - \frac{1}{2}\left.\frac{d^2\langle V\rangle}{dp^2}\right|_{p=p_0}\frac{\partial p'^2}{\partial t} = 0,$$

$$\frac{\partial u}{\partial t} + \langle V\rangle_0\frac{\partial p'}{\partial x} = 0.$$

The relationship $u\frac{\partial p'}{\partial x} = p'\frac{\partial u}{\partial x}$ was used in the derivation of the first equation. This relationship is valid to the adopted accuracy of $O(p'^2)$ and follows from expression (16). The evolutionary equation for a single variable takes the form

$$\langle V\rangle_0^2\frac{\partial^2 p'}{\partial x^2} - \left\langle\frac{V^2}{c^2}\right\rangle_0\frac{\partial^2 p'}{\partial t^2} + \frac{1}{2}\left.\frac{d^2\langle V\rangle}{dp^2}\right|_{p=p_0}\frac{\partial^2 p'^2}{\partial t^2} = 0. \quad (17)$$

Below, we omit the index 0 denoting the unperturbed state. Let us treat the waves propagating in one direction. With the adopted accuracy, we have

$$-\frac{\sqrt{\langle V^2/c^2\rangle}}{\langle V\rangle c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \longrightarrow 2\frac{\partial}{\partial x}$$

(e.g., see Section 93 in [32]). Therefore, after factorization of expression (17) we obtain an evolutionary equation in the Eulerian coordinates,

$$\frac{\partial p'}{\partial t} + c_{\mathrm{eff}}\frac{\partial p'}{\partial x} + \frac{1}{2}\langle V\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2}\frac{d^2\langle V\rangle}{dp^2}p'\frac{\partial p'}{\partial x} = 0. \quad (18)$$

The nonlinear coefficient α_p conditioned by the medium structure can be represented in the following form for the case $c \neq f(p)$:

$$\alpha_p \equiv \frac{1}{2}\langle V\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2}\frac{d^2\langle V\rangle}{dp^2} = \frac{d(u + c_{\mathrm{eff}})}{dp}$$
$$= \langle V\rangle\left\langle\frac{V^3}{c^4}\right\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2},$$

and it is always true that $\alpha_p > 0$. In the case of a homogeneous medium, we have $\frac{dc}{dp} = 0$ and $\alpha_{p\,\mathrm{hom}} = V/c$ [32].

Media with the quantity V/c^2 being constant within the period present a special case. Under the effect of wave perturbation, they behave as homogeneous media. Single structural elements respond to pressure variations in such way that the relative structure does not change, i.e., the ratio $V(\xi, p)/V(\xi, p_0)$ does not depend on ξ. The effective sound velocity is an average characteristic, $c_{\mathrm{eff}} = \sqrt{\langle c^2\rangle}$, in this case. Therefore, the whole set of equations can be expressed in the average variables p, u, $\langle V\rangle$, and $c_{\mathrm{eff}} = \sqrt{\langle c^2\rangle}$. In the case of such media, inhomogeneities do not introduce additional nonlinearity. In terms of the propagation of wave perturbations, such media behave as homogeneous ones.

Let us take a medium where sound velocity in each component is independent of pressure ($c \neq f(p)$) as an example and demonstrate that, in the general case, the structure of the medium gives rise to additional nonlinearity. We consider the ratio of nonlinear coefficients for a structured medium and a homogeneous medium and assume that these media are matched according to conditions (14) and (15). In the space of dimensionless normalized variables, this implies that, for the media under comparison, we have $\langle V\rangle_0 = 1$ and $\langle V^2/c^2\rangle_0 = 1$ at $p = p_0$. Therefore, we obtain

$$\frac{\alpha_p}{\alpha_{p\,hom}} = \langle V\rangle\left\langle\frac{V^3}{c^4}\right\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-2} \geq 1. \quad (19)$$

This inequality is the Cauchy–Schwartz inequality (see formulas (4.6–60) and (15.2–3) and Section 14.2–6 in [33]). Taking into account the fact that $V \geq 0$ and $V/c^2 \geq 0$, we prove expression (19):

$$\langle V\rangle\langle V^3/c^4\rangle \equiv \int_{-\infty}^{\infty} V d\xi \int_{-\infty}^{\infty}\frac{V^3}{c^4}d\xi = \int_{-\infty}^{\infty}\frac{V^2}{c^2}\left(\frac{V}{c^2}\right)^{-1}d\xi$$
$$\times\int_{-\infty}^{\infty}\frac{V^2}{c^2}\frac{V}{c^2}d\xi \geq \left(\int_{-\infty}^{\infty}\sqrt{\frac{V^2}{c^2}\left(\frac{V}{c^2}\right)^{-1}}\sqrt{\frac{V^2}{c^2}\frac{V}{c^2}}d\xi\right)^2$$
$$= \left(\int_{-\infty}^{\infty}\frac{V^2}{c^2}d\xi\right)^2 \equiv \langle V^2/c^2\rangle^2.$$

Now we have only to determine the conditions at which the equality sign is valid. For this purpose, we use the vector form of the Cauchy–Schwartz inequality (see formula (14.2–5) in [33]):

$$|(\mathbf{a}, \mathbf{b})|^2 \le (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}).$$

The equality sign is realized if and only if the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly related: $\mathbf{a} = k\mathbf{b}$ (k = const). In our case, this means that

$$\sqrt{\frac{V^2}{c^2}\left(\frac{V}{c^2}\right)^{-1}} \Big/ \sqrt{\frac{V^2}{c^2}\frac{V}{c^2}} = \text{const}.$$

Here, the equality sign is realized if and only if V/c^2 = const. Such a structured medium was considered above. In the case of all other media with microstructure with the quantity V/c^2 varying within the period, the inequality is true. We arrive at the result that α_p in a structured medium is always greater than $\alpha_{p\,\mathrm{hom}}$ in a homogeneous medium.

Thus, a rigorous consideration of the medium structure reveals nonlinear effects directly conditioned by the inhomogeneity. It is shown that, in the general case, the medium structure introduces additional nonlinearity. In our previous papers [27, 30] we used this effect to develop the mathematical foundations for a new method of diagnostics. According to this method, the properties of a multicomponent medium can be determined with the help of long nonlinear waves propagating in it. The nature of the long-wave interaction is different at different scales. The behavior of the medium at the microscopic level is governed by thermodynamic laws, while at the macroscopic level, it manifests itself as a wave motion for the average characteristics.

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