NONLINEAR PROCESSES

SPECIAL FORM OF THE SINGULAR FUNCTION FOR THE CONTINUOUS PART OF SPECTRAL DATA IN THE INVERSE SCATTERING METHOD

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We describe a procedure for using the method of inverse scattering transform to find the solutions of the Vakhnenko–Parkes equation that are associated with the continuous part of spectral data for the spectral problem. The suggested special form of the singular function gives rise to the periodic solutions. The interaction of N periodic waves is studied. In the general case, the solutions are complex functions. For $N = 1$ and $N = 2$, the real solutions are selected.

1. Introduction

It is significant to look for exact solutions for nonlinear evolution equations in many applications of physics and technology. Various effective approaches have been developed to construct the exact wave solutions of completely integrable equations. One of the fundamental direct methods is undoubtedly the Hirota bilinear method [1, 2], which possesses significant features that make it practical for the determination of multiple soliton solutions. However, this method can be applied only for finding the solitary wave solutions. In this sense, the inverse scattering transform method is more general, although its employment is a fairly difficult procedure [3–5].

This paper deals with the nonlinear evolution equation

$$W_{XXT} + (1 + W_T)W_X = 0. \qquad (1.1)$$

This equation arises from the Vakhnenko equation [2, 6–8]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0 \qquad (1.2)$$

through the transformation [9, 10]

$$u(x, t) = U(X, T) = W_X(X, T),$$

$$x = x_0 + T + W(X, T), \qquad t = X. \qquad (1.3)$$

Equations (1.1) and (1.2) were used for the modeling of high-frequency perturbations in a relaxing medium [8, 11] as well as small-amplitude long waves in rotating fluids of finite depth under the assumption of low-frequency dispersion [12]. Following works [13–15], the equation (1.1) is referred hereafter as the Vakhnenko–Parkes equation (VPE).

The inverse scattering transform method is the most appropriate way of tackling the initial value problem. Recently, this method has been applied to obtain the exact N-soliton solution of the VPE [16]. In this paper, we use the inverse scattering transform method to study the periodic solutions of VPE (1.1) associated with the continuous part of spectral data.

2. The Spectral Problem for the VPE

In order to use the inverse scattering transform method, one has firstly to formulate the associated eigenvalue problem. In [16], it was shown that the pair of equations

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \qquad (2.1)$$

$$3\psi_{XT} + (W_T + 1)\psi = 0 \qquad (2.2)$$

is associated with VPE (1.1) considered here. Note that the inverse scattering problem is related to a third-order

spectral equation (2.1). The inverse problem for third-order spectral equations was considered by Caudrey [17] and Kaup [18]. We adapt the results obtained by these authors to the present problem and describe a procedure for using the inverse scattering transform method to find the solutions of the VPE that are associated with the continuous part of spectral data.

We follow the general theory of the inverse scattering problem for N spectral equations which was developed by Caudrey in [17]. According to [17], the spectral equation (2.1) can be rewritten in the form [16]

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X,\zeta)] \cdot \boldsymbol{\psi} \qquad (2.3)$$

with

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -W_X & 0 \end{pmatrix}. \qquad (2.4)$$

The matrix $\mathbf{A}$ has the eigenvalues $\lambda_j(\zeta)$ and the left- and right-eigenvectors $\tilde{\mathbf{v}}_j(\zeta)$ and $\mathbf{v}_j(\zeta)$, respectively. In the case considered here, we define

$$\lambda_j(\zeta) = \omega_j\zeta, \qquad \lambda_j^3(\zeta) = \lambda, \qquad \mathbf{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \end{pmatrix},$$

$$\tilde{\mathbf{v}}_j(\zeta) = \begin{pmatrix} \lambda_j^2 & \lambda_j & 1 \end{pmatrix}, \qquad (2.5)$$

where $\omega_j = e^{2\pi i(j-1)/3}$ are the cube roots of 1.

The solution of the linear equation (2.1) or, equivalently, Eqs. (2.3) was obtained by Caudrey [17] in terms of the Jost functions $\phi_j(X,\zeta)$ which have the asymptotic behavior

$$\boldsymbol{\Phi}_j(X,\zeta) := \exp\{-\lambda_j(\zeta)X\}\phi_j(X,\zeta) \to \mathbf{v}_j(\zeta),$$

as $X \to -\infty.$ $\qquad (2.6)$

Here, T is regarded as a parameter; the T-evolution of the scattering data will be taken into account later. The solution of the direct problem is given by the system of equations (4.5) in [17]. Since there is a set of symmetry properties such as (6.14) and (6.15) in [17] for the Jost functions $\phi_j(X,\zeta)$, which are also valid in our case, we

need only to consider the element $\phi_1(X,\zeta)$ (as well as $\Phi_1(X,\zeta)$). In the general case, it is necessary to take both the bound-state and continuous spectra into account. According to relation (6.20) in [17], the solution of (2.3) is as follows:

$$\Phi_1(X,\zeta) = 1-$$

$$-\sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)}\Phi_1(X,\omega_j\zeta_1^{(k)})+$$

$$+\frac{1}{2\pi i}\int\sum_{j=2}^{3}Q_{1j}(\zeta')\frac{\exp\{[\lambda_j(\zeta') - \lambda_1(\zeta')]X\}}{\zeta' - \zeta}\times$$

$$\times\Phi_1^{\pm}(X,\omega_j\zeta')d\zeta'. \qquad (2.7)$$

Equation (2.7) involves the spectral data, namely, K poles and the quantities $\gamma_{1j}^{(k)}$ for the bound-state spectrum, as well as the functions $Q_{1j}(\zeta')$ for the continuous spectrum, along all the boundaries of integration, where $\mathrm{Re}(\lambda_1 - \lambda_j) = 0$ over all $j \neq 1$.

The bound-state spectrum is associated with soliton solutions; then it should be $Q_{1j}(\zeta) \equiv 0$ in (2.7). The procedure for finding the exact N-soliton solution of the VPE via the inverse scattering transform method was described in [16]. In the next section, we study the solutions of the VPE, which follow from the continuous part of spectral data.

3. Solutions Associated with the Continuous Part of Spectral Data

We now consider only the continuous spectrum of the associated eigenvalue problem. To do it, we need to put $\gamma_{1j}^{(k)} \equiv 0$ in Eq. (2.7). At each fixed $j \neq 1$, the functions $Q_{1j}(\zeta')$ characterize the singularity of $\Phi_1(X,\zeta)$. This singularity can appear only on boundaries between the regular regions on the ζ-plane. The condition $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ constitutes these boundaries [17]. According to [17], we find that, for $\Phi_1(X,\zeta)$, the complex ζ-plane is divided into four regions by two lines

(i)$\zeta' = \omega_2\xi,$ with $Q_{12}^{(1)}(\zeta') \neq 0,$ $Q_{13}^{(1)}(\zeta') \equiv 0,$

(ii)$\zeta' = -\omega_3\xi,$ with $Q_{12}^{(2)}(\zeta') \equiv 0,$ $Q_{13}^{(2)}(\zeta') \neq 0,$

$$(3.1)$$

where ξ is real. Analysis shows that the direction of the integration in (2.7) is to be such that ξ sweeps from $-\infty$ to $+\infty$.

Let us consider the singular functions $Q_{1j}(\zeta')$ on boundaries, on which the Jost function $\phi_1(X,\zeta)$ is singular, in the special form ($n = 1, 2, ..., N$)

$$
\left.
\begin{aligned}
Q_{12}^{(1)}(\zeta') &= \\
&= -2\pi\mathrm{i} \sum_{n=1}^{N} q_{12}^{(2n-1)} \delta(\zeta' - \zeta'_{2n-1}) \\
Q_{13}^{(1)}(\zeta') &= \\
&= -2\pi\mathrm{i} \sum_{n=1}^{N} q_{13}^{(2n-1)} \delta(\zeta' - \zeta'_{2n-1}) \equiv 0
\end{aligned}
\right\} \text{ on line } \zeta' = \omega_2 \xi,
$$

$$
\left.
\begin{aligned}
Q_{12}^{(2)}(\zeta') &= \\
&= -2\pi\mathrm{i} \sum_{n=1}^{N} q_{12}^{(2n)} \delta(\zeta' - \zeta'_{2n}) \equiv 0 \\
Q_{13}^{(2)}(\zeta') &= \\
&= -2\pi\mathrm{i} \sum_{n=1}^{N} q_{13}^{(2n)} \delta(\zeta' - \zeta'_{2n})
\end{aligned}
\right\} \text{ on line } \zeta' = -\omega_3 \xi.
$$

$$(3.2)$$

At singularity (3.2), relation (2.7) is reduced to the form

$$\Phi_1(X,\zeta) = 1 -$$

$$- \sum_{l=1}^{2N} \sum_{j=2}^{3} q_{1j}^{(l)} \frac{\exp\{[\lambda_j(\zeta'_l) - \lambda_1(\zeta'_l)]X\}}{\zeta'_l - \zeta} \Phi_1(X, \omega_j \zeta'_l). \quad (3.3)$$

Since

$$\phi_{1X}(X,\zeta) = \frac{\mathrm{i}}{\sqrt{3}}[\phi_{1X}(X, -\omega_2\zeta)\phi_1(X, -\omega_3\zeta) -$$

$$-\phi_{1X}(X, -\omega_3\zeta)\phi_1(X, -\omega_2\zeta)] \quad (3.4)$$

takes place [16], the singularities in the form (3.2) appear in pairs $\zeta'_{2n-1} = \omega_2 \xi_n$, $\zeta'_{2n} = -\omega_3 \xi_n$. Considering the limits $\zeta \to \zeta'_l$ and $X \to -\infty$, relation (3.4) immediately yield

$$q_{12}^{(2n-1)} \omega_2 = q_{13}^{(2n)}, \quad \text{with} \quad n = 1, 2, ..., N. \quad (3.5)$$

By expanding $\Phi_1(X,\zeta)$ in an asymptotic series in $\lambda_1^{-1}(\zeta)$, one can obtain (see Eq. (5.11) in [16])

$$\Phi_1(X,\zeta) = 1 - \frac{1}{3\lambda_1(\zeta)}[W(X) - W(-\infty)] + O(\lambda_1^{-2}(\zeta)). \quad (3.6)$$

On the other hand, by defining

$$\Psi_l(X) = \sum_{j=2}^{3} q_{1j}^{(l)} \exp\{\lambda_j(\zeta'_l)X\}\Phi_1(X, \omega_j\zeta'_l), \quad (3.7)$$

we may rewrite relation (3.3) as

$$\Phi_1(X,\zeta) = 1 - \sum_{l=1}^{2N} \frac{\exp\{-\lambda_1(\zeta'_l)X\}}{\zeta'_l - \zeta} \Psi_l(X). \quad (3.8)$$

From (3.6) and (3.8), it can be shown that (see also Eq. (6.38) in [17])

$$W(X) - W(-\infty) = -3 \sum_{l=1}^{2N} \exp\{-\lambda_1(\zeta'_l)X\}\Psi_l(X) =$$

$$= 3\frac{\partial}{\partial X} \ln(\det M(X)). \quad (3.9)$$

The matrix $M(X)$ is defined as follows:

$$M_{lm}(X) = \delta_{lm} - \sum_{j=2}^{3} q_{1j}^{(l)} \frac{\exp\{[\lambda_j(\zeta'_l) - \lambda_1(\zeta'_m)]X\}}{\zeta'_m - \omega_j\zeta'_l}. \quad (3.10)$$

Now, let us consider the T-evolution of the spectral data. By analyzing the solution of Eq. (2.2) as $X \to -\infty$, we find that $\phi_j(X, T, \zeta) = \exp[-(3\lambda_j(\zeta))^{-1}T]\phi_j(X, 0, \zeta)$. Hence, the T-evolution of the scattering data is given by the relations (with $l = 1, 2, ..., 2N$)

$$\lambda_j(T) = \lambda_j(0),$$

$$q_{ij}^{(l)}(T) = q_{ij}^{(l)}(0) \exp\{[-(3\lambda_j(\zeta'_l)^{-1} + (3\lambda_1(\zeta'_l))^{-1}]T\}. \quad (3.11)$$

The final result, including the T-evolution, for the solution of the VPE, which is connected with the continuous part of spectral data, is as follows:

$$U(X,T) = W_X(X,T) = 3\frac{\partial^2}{\partial X^2} \ln(\det M(X,T)). \quad (3.12)$$

Here, $M(X,T)$ is

$$M_{lm} = \delta_{lm} - \sum_{j=2}^{3} q_{1j}^{(l)}(0)(\exp\{[-(3\lambda_j(\zeta'_l)^{-1} + (3\lambda_1(\zeta'_l))^{-1}] \times$$

$$\times T + [\lambda_j(\zeta'_l) - \lambda_1(\zeta'_m)]X\})/(\zeta'_m - \omega_j\zeta'_l), \quad (3.13)$$

and, for all $n = 1, 2, ..., N$,

$$\lambda_1\left(\zeta'_{2n-1}\right) = \omega_2\xi_{2n-1}, \quad \lambda_2\left(\zeta'_{2n-1}\right) = \omega_3\xi_{2n-1}, \quad q_{12}^{(2n-1)} = \omega_2\beta_{2n-1}, \quad q_{13}^{(2n-1)} = 0,$$
$$\lambda_1\left(\zeta'_{2n}\right) = -\omega_3\xi_{2n-1}, \quad \lambda_3\left(\zeta'_{2n}\right) = -\omega_2\xi_{2n-1}, \quad q_{12}^{(2n)} = 0, \qquad\qquad q_{13}^{(2n)} = \omega_3\beta_{2n-1}.$$

As will be clear from the examples in the next section, solution (3.12), (3.13) contains N frequencies from the continuous spectrum. Thus, the singularities in the form (3.2) constitute N frequencies from the continuous part of spectral data. For this reason, solution (3.12), (3.13) will be referred to as the N-mode solution of the VPE. It is clear that the N-mode solution describes the interaction of N periodic waves. For the N-mode solution, there are N arbitrary constants ξ_i and N arbitrary constants β_i. The constants ξ_i are real values, while the constants β_i, in the general case, are complex values.

The solution obtained through matrix (3.13) is, in general, the complex function. Since we are interested in the real solutions for the VPE, we have to restrict the arbitrariness in choice of the constants β_i. For one- and two-mode solutions, we succeeded in finding these restrictions.

4. Examples of One- and Two-mode Solutions

In order to obtain the one-mode solution of VPE (1.1), we need firstly to calculate the 2×2 matrix M according to (3.13) with $N = 1$. We find that the matrix is

$$\begin{pmatrix} 1 - \dfrac{i\omega_2\beta_1}{\sqrt{3}\xi_1}\exp[-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T] & -\dfrac{\omega_3\beta_1}{2\xi_1}\exp[2\omega_3\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T \\[2ex] \dfrac{\omega_2\beta_1}{2\xi_1}\exp[-2\omega_2\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T] & 1 - \dfrac{i\omega_3\beta_1}{\sqrt{3}\xi_1}\exp[-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T] \end{pmatrix}, \tag{4.1}$$

and its determinant is

$$\det M(X,T) = \left[1 + c_1\exp(-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T)\right]^2,$$

$$c_1 = \frac{i\beta_1}{2\sqrt{3}\xi_1}. \tag{4.2}$$

Thus, as is already noted, the singularity in the form (3.2) is responsible for one frequency from the continuous part of spectral data for the one-mode solution (3.9).

The condition for W to be real requires the restriction on the constant β_1 (if the constant ξ_1 is arbitrary). We succeeded in obtaining this restriction. Namely, the constant $c_1 = |c_1|\exp(i\chi_1)$, which is complex-valued in the general case, should satisfy the relation $|c_1| = 1$, whereas the arbitrary real constant χ_1 defines a initial shift of the solution $X_1 = \chi_1/(\sqrt{3}\xi_1)$. Then we have

$$\det M(X,T) = \left[1 + \exp\left(-i\sqrt{3}\xi_1(X - X_1) + \frac{T}{i\sqrt{3}\xi_1}\right)\right]^2. \tag{4.3}$$

The final result for one mode of the continuous spectrum is solution (3.9) with (4.3). Namely,

$$W(X,T) =$$

$$= -3\sqrt{3}\,\xi_1\tan\left(\frac{\sqrt{3}}{2}\xi_1(X - X_1) + \frac{T}{2\sqrt{3}\xi_1}\right) + \text{const} \tag{4.4}$$

or

$$U(X,T) = W_X(X,T) =$$

$$= -\frac{9}{2}\,\xi_1^2\cos^{-2}\left(\frac{\sqrt{3}}{2}\xi_1(X - X_1) + \frac{T}{2\sqrt{3}\xi_1}\right). \tag{4.5}$$

The same solutions have been obtained recently by different direct methods, for example, by the (G'/G)-expansion method [15]. However, only the notation of the solution in the form (3.9), (3.13) enables us to study the interaction of two periodic waves.

Let us now consider the two-mode solution of the VPE. In this case, $M(X,T)$ is a 4×4 matrix. We will not give the explicit form here, but we find that

$$\det M = (1 + q_1 + q_2 + bq_1q_2)^2, \qquad (4.6)$$

where

$$q_i = c_i \exp[-i\sqrt{3}\xi_i X + (i\sqrt{3}\xi_i)^{-1}T],$$

$$c_i = \frac{i\beta_i}{2\sqrt{3}\xi_i}, \quad b = \left(\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1}\right)^2 \frac{\xi_1^2 + \xi_2^2 - \xi_1\xi_2}{\xi_1^2 + \xi_2^2 + \xi_1\xi_2}. \qquad (4.7)$$

Since the quantity W should be real, and the constants ξ_i are arbitrary, but real, there is the restriction on the constants $c_i = |c_i|\exp(i\chi_i)$. The real constants χ_i define the initial shifts of the solutions $X_i = \chi_i/(\sqrt{3}\xi_i)$. The comprehensive analysis shows that the relations $|c_1| = |c_2| = 1/\sqrt{b}$ are the sufficient conditions in order that W take real values. Thus, the interaction of two waves for the VPE is described by relation (3.9) with

$$\det M = \left(1 + \frac{1}{\sqrt{b}}q_1 + \frac{1}{\sqrt{b}}q_2 + q_1q_2\right)^2,$$

$$q_i = \exp\left[-i\sqrt{3}\xi_i(X - X_i) + (i\sqrt{3}\xi_i)^{-1}T\right], \qquad (4.8)$$

and b is as that in (4.7).

It is necessary to note that the solutions constructed by the suggested method are singular ones (see (4.5)). However, the solution obtained here in combination with the developed approach can result in solving the important problem, namely, in analyzing the formation of loop-like solitonic solutions inherent to Eq. (1.2), as well as in the physical interpretation of these ambiguous solutions.

5. Conclusion

The inverse scattering transform method is applied to the Vakhnenko–Parkes equation for finding the solutions that are associated with both bound-state and continuous spectra of the spectral problem. The special form of the singular function is suggested to obtain the periodic solutions, which are, in the general case, the complex functions. For $N = 1$ and $N = 2$, the real periodic solutions are selected.

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ОСОБЛИВИЙ ВИГЛЯД СИНГУЛЯРНОЇ ФУНКЦІЇ ДЛЯ НЕПЕРЕРВНОЇ ЧАСТИНИ СПЕКТРАЛЬНИХ ДАНИХ У МЕТОДІ ОБЕРНЕНОЇ ЗАДАЧІ РОЗСІЯННЯ

В.О. Вахненко

Р е з ю м е

У роботі описано процедуру оберненої задачі розсіяння для знаходження розв'язків рівняння Вахненка–Паркеса, які пов'язуються з неперервною частиною спектральних даних у асоційованій спектральній задачі. Запропоновано особливий вигляд сингулярної функції, який приводить до періодичних розв'язків. Вивчено взаємодію N періодичних хвиль. У загальному випадку розв'язки – комплексні функції. Для $N = 1$ та $N = 2$ виділено дійсні розв'язки.