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ASYMPTOTIC JUSTIFICATION OF THE LYAKHOV MODEL FOR MULTICOMPONENT MEDIA

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The Lyakhov model for multicomponent media, whose rigorous mathematical justification is obtained using an asymptotic averaging technique is considered. It is shown that the nature of long-wave action differs at different scale levels. At the microlevel the behavior of the medium obeys thermodynamic laws, while at the macrolevel it manifests itself as wave motion for average characteristics.

The Lyakhov model has been used for several decades to describe wave processes in explosions in the ground and porous multicomponent media [1-3]. But the fact that the model uses the equations of motion of a homogeneous medium without rigorous proof necessitates its justification.

The Lyakhov Model. For definiteness, let us analyze the Lyakhov model [3] using as an example water-saturated ground containing three components: air ($i = 1$), water ($i = 2$), and solid matter ($i = 3$). The components are considered barotropic. We consider only long-wave disturbances, i.e., the characteristic size of the microstructure ε' is considered small compared with the wavelength λ. In this model it is assumed *a priori* that one average pressure p_L, one velocity u_L, and one specific volume v_L are achieved in each macrovolume. The equations of motion of the ground are written as for a homogeneous medium. In a Lagrangian system of coordinates (l, t) for one-dimensional motion they are written as

$$\frac{\partial u_L}{\partial t} + v_{L0}\left[\frac{r}{l}\right]^{\nu-1}\frac{\partial p_L}{\partial l} = 0, \qquad \frac{\partial v_L}{\partial t} - \nu v_{L0}\,\frac{\partial r^{\nu-1}u_L}{\partial l^{\nu}} = 0, \tag{1}$$

where r is the Eulerian space coordinate; $\nu = 1$, 2, or 3 for plane, cylindrical, or spherical symmetry respectively; subscript 0 denotes the initial state.

Equations (1) are supplemented by the equations of state [3]

$$\frac{\dot{v}_L}{v_{L0}} = \varphi(p_L)\dot{p}_L - \frac{\alpha_1}{\eta}\,\psi(p_L, v_L), \tag{2}$$

$$\varphi(p_L) = -\sum_{i=2}^{3}\frac{\alpha_i}{\rho_{i0}c_{i0}^2}\left[\frac{\gamma_i(p_L-p_0)}{\rho_{i0}c_{i0}^2}+1\right]^{-(1+\gamma_i)/\gamma_i}, \tag{3}$$

$$\psi(p_L, v_L) = p_L - p_0 - \frac{\rho_{10}c_{10}^2}{\gamma_1}\left[\left\{\frac{v_L}{v_{L0}} - \sum_{i=2}^{3}\alpha_i\left[\frac{\gamma_i(p_L-p_0)}{\rho_{i0}c_{i0}^2}+1\right]^{-1/\gamma_i}\right\}^{-\gamma_1}\alpha_1^{\gamma_1} - 1\right]. \tag{4}$$

Here α_i, $\rho_{i0} \equiv v_{i0}^{-1}$, and c_{i0} are the volume concentration, density, and sound speed of the ith component with the initial pressure p_0; γ_i is the adiabatic exponent in the equation of state of the ith component; η is the coefficient of volume viscosity of the ground.

System (1)-(4) describes the evolution of long-wave disturbances in multicomponent media in a hydrodynamic approximation. Equation of state (2) follows from semi-empirical considerations of the medium

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and the properties of its components [3]. The solid matter and the liquid are described by the Tate equilibrium equation, and the gaseous component, by a dynamic equation of state that takes into account the strain rate of the ground. Following [3], at the initial moment of loading the gas is incompressible. This means that the frozen sound speed c_{f1} of the gaseous component in the ground is an infinite quantity:

$$c_{f1} = \infty. \tag{5}$$

The low-frequency sound speed in the gaseous component c_{e1} is equal to the sound speed in air.

Asymptotic Averaged Model. There is a class of mathematically rigorous asymptotic methods for averaged description of materials of a regular microstructure, including the asymptotic averaging technique [4, 5]. It is used to simulate long waves in compressible media [6]. In contrast to the well-known results for the plane case [6–9], we derive an averaged system of equations for one-dimensional motions cf any symmetry.

The basic hydrodynamic equations for unsteady one-dimensional motions of the medium should be written in the Lagrange variables. In this system the microstructure does not change with wave motions. The initial equations are the equations of motion for an individual component:

$$\frac{\partial r^\nu}{\partial l^\nu} = \frac{v}{v_0}, \qquad u = \frac{\partial r}{\partial t}, \qquad \frac{\partial u}{\partial t} + v_0 \left(\frac{r}{l}\right)^{\nu - 1} \frac{\partial p}{\partial l} = 0. \tag{6}$$

The continuity equation can be written in alternative form:

$$\frac{\partial v}{\partial t} - \nu v_0 \frac{\partial r^{\nu-1} u}{\partial l^\nu} = 0. \tag{7}$$

Each component in an elementary cell of the medium is considered relaxing and is described by the dynamic equation of state [8, 9]

$$d\rho = c_f^{-2} dp - \tau_r^{-1} (\rho - \rho_e) dt, \tag{8}$$

were c_f is the frozen sound speed; τ_r is the characteristic relaxation time of an element of the medium. The equilibrium density ρ_e is related to pressure by the equilibrium equation of state

$$\rho_e - \rho_0 = \int_{p_0}^{p} c_e^{-2}\, dp,$$

where c_e is the equilibrium sound speed.

For a rigorous mathematical derivation of the averaged system of equations let us assume that the medium is layered and has a periodic structure. Naturally, in the case of cylindrical and spherical symmetries, the layers have the appropriate symmetry.

We consider long-wave disturbances and impose the conditions of consistency of mass velocities and stresses at the layer interfaces:

$$[u] = 0, \qquad [p] = 0. \tag{9}$$

It is convenient to use dimensionless variables [10], for which the resulting dimensionless equations will not differ in form from the initial equations. Therefore, we consider relations (6) and (7) to be written in dimensionless variables.

According to the asymptotic averaging technique [4, 5], the spatial mass coordinate $m = l^\nu / v_0$ decomposes into slow (s) and fast (ξ) independent variables:

$$m = s + \varepsilon \xi, \qquad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1} \frac{\partial}{\partial \xi}.$$

The dimensionless period of the structure $\varepsilon = \varepsilon' / \lambda$ is a small parameter. The solutions r^ν, p, u, and v are sought in the form of series in powers of ε:

$$r^\nu(m,t) = (r^\nu)^{(0)}(s,t,\xi) + \varepsilon (r^\nu)^{(1)}(s,t,\xi) + \varepsilon^2 (r^\nu)^{(2)}(s,t,\xi) + \dots,$$
$$v(m,t) = v^0(s,t,\xi) + \varepsilon v^{(1)}(s,t,\xi) + \varepsilon^2 v^{(2)}(s,t,\xi) = \dots .$$

The functions $v^{(i)}(\xi)$ are considered i-periodic in ξ.

Following the procedure described in detail for the plane case in [9], for the order $o(\varepsilon^{-1})$ we obtain

$$\frac{\partial (r^\nu)^{(0)}}{\partial \xi} = 0, \qquad \frac{\partial p^{(0)}}{\partial \xi} = 0, \qquad \frac{\partial (r^\nu)^{(0)} u^{(0)}}{\partial \xi} = 0. \tag{10}$$

Consequently, the mass velocity $u^{(0)}$, pressure $p^{(0)}$, and Eulerian coordinate $(r^\nu)^{(0)}$ are independent of the fast variable ξ.

For the order $o(\varepsilon^0)$ we have

$$\frac{\partial u^{(0)}}{\partial t} + \nu (r^\nu)^{(0)} \frac{\partial p^{(0)}}{\partial s} + \nu (r^\nu)^{(1)} \frac{\partial p^{(0)}}{\partial \xi} + \nu (r^\nu)^{(0)} \frac{\partial p^{(1)}}{\partial \xi} = 0. \tag{11}$$

Since $\langle \partial p^{(1)} / \partial \xi \rangle = 0$ in view of the periodicity of $p^{(1)}$ with respect to ξ, averaging over a period in the Lagrangian mass coordinates $\langle \cdot \rangle = \int d\xi$, we have one of the desired equations:

$$\frac{\partial u^{(0)}}{\partial t} + \nu (r^\nu)^{(0)} \frac{\partial p^{(0)}}{\partial s} = 0. \tag{12}$$

On the other hand, we obtain $\partial p^{(1)}/\partial \xi = 0$ by subtracting Eq. (12) from (11). Therefore, $p^{(1)}$ is also independent of ξ.

One can easily write the remaining equations in averaged form:

$$\frac{\partial (r^\nu)^{(0)}}{\partial s} = \langle v \rangle, \qquad u^{(0)} = \frac{\partial r^{(0)}}{\partial t}, \qquad \frac{\partial u^{(0)}}{\partial t} + \nu (r^\nu)^{(0)} \frac{\partial p^{(0)}}{\partial s} = 0, \tag{13}$$

$$d\langle v^{(0)} \rangle = -\Big\langle \frac{(v^{(0)})^2}{c_f^2} \Big\rangle dp - \Big\langle \frac{(v^{(0)})^2}{\tau_r c_e^2} (p^0 - p_e^{(0)}) \Big\rangle dt. \tag{14}$$

Equality (7) has the form

$$\frac{\partial \langle v^{(0)} \rangle}{\partial t} - \nu \frac{\partial (r^\nu)^{(0)} u^{(0)}}{\partial s} = 0. \tag{15}$$

Unlike the quantities $u^{(0)}$, $p^{(0)}$, $p^{(1)}$, and $r^{(0)}$, the specific volume $v^{(0)}$ is a function of ξ. System (13)–(15) is integrodifferential. Below, we shall restrict ourselves to a zeroth approximation with respect to ε and omit the superscript 0.

It should be noted that the load at the level of the medium's microstructure is wave-free, since pressure and mass velocity are independent of ξ. The behavior of the medium at this level obeys only thermodynamic laws. At the macrolevel, the state of the medium is described by the laws of wave dynamics for average characteristics. At this level of hierarchy, Eqs. of motion (13) and (15) remain unchanged if one changes the arrangement of the layers in an elementary cell or splits them. Thus, relations (13)–(15) describe identically the behavior of any quasi-periodic (statistically nonuniform) medium that has the same mass content of components at the microstructure level regardless of the matter location in the cell.

Comparison of the Models. Let us compare Eqs. (1) and (2) of the Lyakhov model with the averaged Eqs. (13)–(15). In the general case, an elementary cell can contain a set of many layers of different components. In a layered medium the specific volume distribution $v = v(\xi)$ in an elementary cell is a step function and hence

$$\langle v \rangle \equiv \int_0^1 v(\xi) d\xi = \sum_{i=1}^{3} \beta_i v_i, \tag{16}$$

where β_i is the size of the component in a period in the scale of the fast variable. The quantity β_i is the mass content of the component and a structural characteristic that does not change with wave motions.

In the Lyakhov model for a specific volume, formula (3.3) from [3] is valid:

$$\frac{v_L}{v_{L0}} = \sum_{i=1}^{3} \alpha_i \frac{v_i}{v_{i0}}. \tag{17}$$

One can easily see that there is a relationship between α_i and β_i:

$$\beta_i = \alpha_i \frac{v_{L0}}{v_{i0}}. \tag{18}$$

Hence it follows that in the asymptotic averaged model the specific volume (16) averaged over the structure period is the specific volume (17) defined by Lyakhov as

$$v_L = v_{L0} \sum_{i=1}^{3} \alpha_i \frac{v_i}{v_{i0}} = \sum_{i=1}^{3} \alpha_i \frac{v_{L0}}{v_{i0}} v_i = \sum_{i=1}^{3} \beta_i v_i = \langle v \rangle.$$

In averaged Eqs. (13) and (15) the pressure p and the mass velocity u remain unchanged over the structure period. Consequently, Eqs. of motion (1) coincide with (13) and (15). In this case, $s = l^{\nu}/v_{L0}$ should hold. Thus, the assumption that in the Lyakhov model the pressure p_L and the mass velocity u_L are average quantities has received a rigorous proof. Now the subscript L in (1)–(4) can be omitted.

Let us compare Eqs. of state (2) and (14). We write (14) in a form that is convenient for comparison:

$$v^{-1}\frac{d\langle v \rangle}{dt} = -v^{-1}\left\langle \frac{v^2}{c_f^2} \right\rangle \frac{dp}{dt} - v^{-1}\left\langle \frac{v^2}{\tau_r c_e^2}(p - p_e) \right\rangle. \tag{19}$$

The left-hand sides of (2) and (19) coincide. Compare the coefficients of dp/dt. The dependence $\varphi(p)$ can be written with allowance for (18) as

$$\varphi(p) = -\sum_{i=2}^{3} \frac{\alpha_i}{\rho_{i0} c_{i0}^2} \left(\frac{v_i}{v_{i0}}\right)^{1+\gamma_i} = -v^{-1} \sum_{i=1}^{3} \beta_i \frac{v_i^2}{c_{i0}^2} \left(\frac{v_i}{v_{i0}}\right)^{\gamma_i^{-1}}.$$

The Lyakhov model uses a definite relationship between the pressure and sound speed components given by the Tate equation. In the asymptotic model this dependence remains arbitrary. If we now use the relation

$$c_{fi} = c_{i0} \left(\frac{\rho_i}{\rho_{i0}}\right)^{\gamma_i^{-1}} \qquad (i = 2,\, 3) \tag{20}$$

and take into account the condition of air incompressibility (5), the coefficients of dp/dt in (2) and (19) coincide.

Let us simplify ψ in (4) using (18):

$$\psi(p, v) = p - p_0 - \frac{\rho_{10} c_{10}^2}{\gamma_1} \left[\left\{ \frac{v}{v_0} - \sum_{i=2}^{3} \alpha_i \frac{v_i}{v_{i0}} \right\}^{-\gamma_1} \alpha_1^{\gamma_1} - 1 \right] = p - p_0 - \frac{\rho_{10} c_{10}^2}{\gamma_1} \left[\left(\frac{v_1}{v_{10}} \right)^{-\gamma_1} - 1 \right] = p - p_0 \left(\frac{v_1}{v_{10}} \right)^{-\gamma_1} = p - p_{e1}.$$

The dependence $p_{e1} = p_{e1}(v_1)$ is the equilibrium equation of state of the air in the ground, i.e., the equation of state of free air. Finally the second term on the right-hand side of (2) becomes

$$(p - p_{e1})\alpha_1/\eta.$$

Now we can easily show that the right terms in (2) and (19) coincide, if we assume that only the gaseous component relaxes, i.e.,

$$c_{f1} \neq c_{e1}, \qquad c_{f2} = c_{e2}, \qquad c_{f3} = c_{e3}, \tag{21}$$

and there is a unique relationship between the relaxation time τ_{r1} and η

$$\tau_{r1} = \eta \frac{v_1^2}{c_{e1}^2 v_{10}}. \tag{22}$$

Equations (2) and (19) coincide if conditions (5) and (20)–(22) are fulfilled.

Conclusion. The asymptotic averaged model (13)–(15) describes waves in media that can have any number of relaxing components [arbitrary distributions of $v(\xi)$ and $\tau_r(\xi)$ in an elementary structure cell], with arbitrary dependences of sound speeds on pressure. A particular case of this model is the Lyakhov model, in which only one (gaseous) component, which is incompressible at the initial moment of loading, relaxes and the sound speeds are definite functions of pressure. The Lyakhov model is of an asymptotic nature.

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