Sov. J. Chem. Phys., Volume 12(3), 1994, pp. 536–546
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AVERAGED DESCRIPTION OF SHOCK-WAVE PROCESSES IN PERIODIC MEDIA*

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Barotropic periodic relaxing media are modeled by averaged simultaneous equations of zeroth order in the period of the structure. Generally, these equations are intergo-differential and cannot be reduced to equations in averaged characteristics. In the case of low-frequency perturbations the equation for the averaged bulk viscosity of such media is deduced and solved to yield a simple wave-type expression. An evolution equation is derived with due regard for weak structure-induced nonlinearity. It is suggested that the medium structure be ascertained by means of moderate pressure waves. The averaged equation in a slowly changing variable is isolated invoking Fourier series and piecewise constant functions of a special form. Numerical algorithm was so elaborated that the integration pitch was not limited by the microstructure size. The results of calculations of self-simulating rarefaction waves are presented.

The modern experimental techniques offer a means for studying the internal structure of a shock-compressed medium. A native medium is a complex inhomogeneous system with hierarchic size-distribution of structural elements [1-4]. Macroscopic modeling presumes averaged description of elements occupying the lower "footsteps of the scale of ranks" [5,6]. The multitude of phenomena occurring at still lower hierarchic stages can be modeled within the framework of the relaxation approach [7].

Propagation of long wavelength perturbations over an inhomogeneous medium is studied using a periodic system as an example. The elements of the microstructure are assumed large to such an extent that they obey the classical laws of continuum mechanics. In the Lagrangian frame of reference, one-dimensional motion of a

structural element is governed by equations

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial m} = 0 , \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial m} = 0 . \qquad (1)$$

Matching is done by equating the mass velocities and the pressures at the element boundaries. The relaxation effects are allowed for by choosing the dynamic equation of state

$$d\rho = c_f^{-2}dp - \tau^{-1}(\rho - \rho_e)dt , \qquad (2)$$

substantiated within the thermodynamics of irreversible processes [7]. Here τ is the relaxation time, and c_e and c_f are the equilibrium and frozen sound speeds. The other notation is conventional. The equilibrium equation of state is assumed known

$$\rho_e - \rho_0 = \int_{p_0}^{p} c_e^{-2}dp ,$$

where subscript "0" refers to the initial state.

When studying waves propagating over periodic media by distances far exceeding the structure period, ε, the equations of motion are virtually not amenable to numerical integration. One way of exploring inhomogeneous media consists in asymptotic averaging of equations with fast-oscillating coefficients [5,6]. According to this method, the spatial coordinate is decomposed into two components, slow s and fast ξ

$$m = s + \varepsilon\xi , \quad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi} .$$

Dependent variables are expanded in a power series in the structure period, ε, e.g.

$$p(m,t) = p^{(0)}(s,t,\xi) + \varepsilon p^{(1)}(s,t,\xi) + \varepsilon^2 p^{(2)}(s,t,\xi) + \ldots$$

Functions $p^{(i)}$ are supposed to be single-period with respect to ξ. The structure period is invariant only in the Lagrangian frame of reference, which permits one to average the above equations of motion. Integrating the equations which involve ε to the zeroth power, over the structure period, one arrives at the averaged set [8,9]

$$\frac{\partial \langle V^{(0)} \rangle}{\partial t} - \frac{\partial u^{(0)}}{\partial s} = 0 , \quad \frac{\partial u^{(0)}}{\partial t} + \frac{\partial p^{(0)}}{\partial s} = 0 , \tag{3}$$

$$\langle V_0 \rangle - \langle V^{(0)} \rangle = \left\langle V_0 \left(\tau c_f^{-2} \frac{dp}{dt} + \int_{p_0}^{p} c_e^{-2} dp \right) \left[\left(1 + \tau \frac{d}{dt} \right) (V^{(0)})^{-1}{}_{*} \right]^{-1} \right\rangle . \tag{4}$$

Here $\langle \ldots \rangle = \int_0^1 (\ldots) d\xi$ symbolizes averaging over the structure period, where ξ is subject to the normalization condition $\int_0^1 d\xi = 1$.

The asterisk in (4) implies that the d/dt operator is open. In what follows, we shall restrict our consideration to the zeroth approximation, and superscript "0" will be omitted.

Pressure p and mass velocity u were proved to be independent of the fast variable ξ, which cannot be said about the specific volume V. The averaged simultaneous equations (3)-(4) are integro-differential, and in the general case cannot be reduced to averaged characteristics.

Now we describe one possible way of so reducing an equation that all the sought-for functions depend only on the slow variable and time [9,10]. The ξ-dependent functions are expanded (e.g., in Fourier series) in terms of basis fast-oscillating functions on an interval equal to the structure period. Invoking equation of state (2) and relationship $\rho V = 1$ facilitates application of the series. Indeed, equation of state (2) represents the sum of the products of no more than two functions depending on ξ. As a result, the initial equations can be reduced to those in coefficients of series, which are the functions of s and t. Schematically, these equations can be specified as

$$(c_f^{-2})_k \dot{p} - \dot{\rho}_k + \sum_{n=-\infty}^{\infty} [(c_e^{-2})_n (\tau^{-1})_{k-n} (p - p_0) + (\tau^{-1})_{k-n} (\rho_{0n} - \rho_n)] = 0 ,$$

$$\sum_{n=-\infty}^{\infty} (\rho_n V_{-n})_k = \delta_{k0} , \quad k = 0, \pm 1, \pm 2, \ldots$$

Here the kth terms are the coefficients of the appropriate expansions of functions ρ, ρ_0, V, V_0, c_f^{-2}, c_e^{-2}, and τ^{-1}. Therefore, the integration step is restricted by the perturbation wavelength rather than by the structure period. Thus, the main computational problem associated with the smallness of the integration step is obviated, and the equations of motion can be solved over a large region of wave propagation within a reasonable time.

If the results of computations are to be compared with the experimental data, the equations are conveniently specified in the Euler coordinates, in which case the microstructure dimensions are variable, and hence, the asymptotic averaging technique proves impractical. Yet, Eqs.(3) derived in the zeroth approximation with respect to ε and expressed in terms of averaged characteristics p, u, and $\langle V \rangle$, can be recast in the Euler coordinates. To do this, we deduce a relationship between independent variables of the Euler (x, t_E) and Lagrangian (s,t) frames of reference

$$x = x(s,t) , \quad t_E = t . \tag{5}$$

Of importance is the fact that in the zeroth approximation with respect to ε the particle velocity is constant throughout the structure period, and hence, it is justifiable to introduce the notion of an average particle trajectory. Coordinate x of the specific particle (i.e. its trajectory) varies with time as

$$(\partial x / \partial t)_s = u(s,t) .$$

Apart from that, s and x vary simultaneously, i.e. transformation (5) can be specified in the following differential form

$$dx = Ads + udt, \quad t_E = t . \tag{6}$$

From physical considerations, the particle location is unambiguously determined by the particle itself and time. Mathematically this implies that dx in (6) is the exact differential, and therefore

$$\partial A/\partial t = \partial u/\partial s .$$

This condition is true if $A = \langle V \rangle$, since in this case it transforms to the continuity equation (3).

Thus, the relationship between the Lagrangian and Euler coordinates reads

$$dx = \langle V \rangle ds + udt , \quad t_E = t .$$

In the Euler frame of reference Eqs.(3) assume the form

$$\frac{\partial \langle V \rangle^{-1}}{\partial t_E} + \frac{\partial u \langle V \rangle^{-1}}{\partial x} = 0 , \quad \frac{\partial u}{\partial t_E} + u\frac{\partial u}{\partial x} + \langle V \rangle \frac{\partial p}{\partial x} = 0 . \tag{7}$$

Taking the size of an elementary cell in the Euler frame of reference as x_1 and invoking the normalization condition, one obtains

$$\langle V \rangle = \int_0^1 V(\xi)d\xi = \int_0^{x_1} V\rho/\varepsilon dx = x_1/\varepsilon = x_1/\int_0^{x_1} \rho dx = \tilde{\rho}^{-1} .$$

Hence, in the Euler frame of reference $\tilde{\rho} = \langle V \rangle^{-1}$ is the mean density. It is this quantity which is measured experimentally. Meanwhile, the mean values of p and u are identical in the both frames of reference.

Now we explore certain "average" properties of acoustic waves propagating over a periodic relaxing medium. The variables will be sought for in the form

$$p = p_0 + p' , \ \rho = \rho_0 + \rho' , \ V = V_0 + V' ,$$

where p', ρ', and V' are the increments in pressure, density, and specific volume in the acoustic wave. In a linear approximation the set of averaged equations (3)-(4) assumes the form

$$\frac{\partial \langle V' \rangle}{\partial t} - \frac{\partial u}{\partial s} = 0 , \quad \frac{\partial u}{\partial t} + \frac{\partial p'}{\partial s} = 0 , \tag{8}$$

$$\langle V' \rangle = -\langle V_0^2 \frac{c_e^{-2} + \tau c_f^{-2} d/dt}{1 + \tau d/dt} \rangle p' . \tag{9}$$

In many instances, small perturbations of pressure propagate similarly in periodic and homogeneous media. Now we analyze some of these instances. In a nonrelaxing periodic medium, equation of state (4) reduces to

$$d\langle V \rangle = -\langle V_0^2/c^2 \rangle dp . \tag{10}$$

Equations (8),(10) are expressed in terms of averaged characteristics. When Eqs.(8,10) are compared with initial equations for homogeneous medium, it is apparent that small perturbations of pressure in periodic and homogeneous nonrelaxing media behave similarly, on average, provided $V_0^2/c^2 = \langle V_0^2/c^2 \rangle$.

In the special case of a periodic medium involving two components, one of which relaxes, small averaged perturbations of pressure propagate in the same way as in a homogeneous relaxing medium, if the consistency conditions, $V_0^2/c_e^2 = \langle V_0^2/c_e^2 \rangle$, $V_0^2/c_f^2 = \langle V_0^2/c_f^2 \rangle$, $\tau = \tau_1$, are true [9].

Moreover, propagation of low-amplitude waves in a periodic medium with a countable number of relaxing components is similar to that in a homogeneous medium with the same number of independent relaxation processes. The similarity of small perturbation propagation over periodic and homogeneous media was verified numerically [8,9].

In what follows, only nonrelaxing systems will be considered, and no limitation will be imposed on the perturbation amplitude. With the effective mean sound speed specified as

$$\tilde{c} = (\langle V \rangle^2/\langle V^2/c^2 \rangle)^{1/2} ,$$

the equation of state transforms to

$$d\tilde{\rho} = \tilde{c}^{-2}dp \ .$$

The above relationship together with equations of motion (3) (or (7)) make up a set of equations in conventional notation. The effective sound speed, c, is different from both the mean velocity $\langle c\rangle$ and $\langle c^2\rangle^{1/2}$, i.e. for nonrelaxing media, too, these equations cannot be expressed in terms of averaged characteristic.

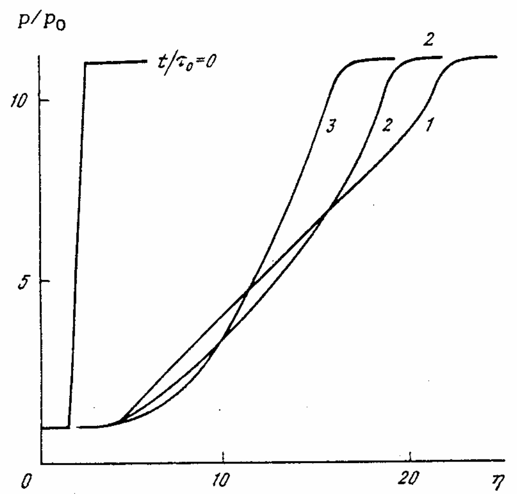

FIGURE 1. Self-simulating rarefaction waves: *1* - homogeneous medium, or $c_1/c_2 = \sqrt{2}$; *2*, *3* - periodic media, $c_1/c_2 = \sqrt{2}/5$ (*2*) and $5\sqrt{2}$ (*3*).

Simultaneous equations (3) and (10) are hyperbolic. In the Lagrangian frame of reference characteristic equations assume the form

$$ds/dt = \pm\langle V^2/c^2\rangle^{-1/2} \ ; \qquad (11)$$

for these equations, quantities

$$I_\pm = u \pm \int\langle V^2/c^2\rangle^{1/2}dp \ , \qquad (12)$$

referred to as Riemann invariants, remain unchanged under transformation of the coordinate system.

The medium structure affects the propagation of a simple high-amplitude wave even in the limit of long wavelength perturbations. Equations (11), (12) involve nonlinear term $\langle V^2/c^2\rangle$ which depends on the pressure and medium structure.

Now we analyze nonlinearity introduced into the wave motion by the medium structure and deduce the evolution equation allowing for a weak nonlinearity. To within the second-order terms, the functional dependence of the mean specific volume on the pressure increment, p', can be represented by the series

$$\langle V\rangle(p) = \langle V_0\rangle + \frac{d\langle V\rangle}{dp}p' + \frac{1}{2}\frac{d^2\langle V\rangle}{dp^2}p'^2 \ .$$

Then the set of equations (7) can be written as (subscripts "0" and "*E*" are omitted)

$$\langle V\rangle\frac{\partial u}{\partial x} + \langle\frac{V^2}{c^2}\rangle\frac{\partial p'}{\partial t} - \frac{1}{2}\frac{d^2\langle V\rangle}{dp^2}\frac{\partial p'^2}{\partial t} = 0 \ , \quad \frac{\partial u}{\partial t} + \langle V\rangle\frac{\partial p'}{\partial x} = 0 \ .$$

The evolution equation in variable p' assumes the form

$$\langle V\rangle^2\frac{\partial^2 p'}{\partial x^2} - \langle\frac{V^2}{c^2}\rangle\frac{\partial^2 p'}{\partial t^2} + \frac{1}{2}\frac{d^2\langle V\rangle}{dp^2}\frac{\partial^2 p'^2}{\partial t^2} = 0 \ . \qquad (13)$$

Consider waves propagating unidirectionally. To within the aforesaid accuracy,

$$\frac{\langle V^2/c^2\rangle^{1/2}}{\langle V\rangle}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} = 2\frac{\partial}{\partial x} \ .$$

Factorization of Eq.(13) yields

$$\left(\frac{\langle V^2/c^2\rangle^{1/2}}{\langle V\rangle}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p' - \frac{1}{2}\langle\frac{V^2}{c^2}\rangle^{-1}\frac{d^2\langle V\rangle}{dp^2}p'\frac{\partial p'}{\partial x} = 0 \ .$$

The nonlinearity factor, α_p, associated with the periodicity of the medium ($c \neq f(p)$) can be specified as

$$\alpha_p = \frac{1}{2}\langle V\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2}\frac{d^2\langle V\rangle}{dp^2} = \frac{d(u + \tilde{c})}{dp} = \langle V\rangle\left\langle\frac{V^3}{c^4}\right\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2} .$$

Here α_p is always > 0, whence, in particular, it follows that no rarefaction shock wave can arise. In a homogeneous medium, dc/dp = 0 and $\alpha_p = V/c$. By the way, media in which V/c^2 remains constant within the structure period behave in the same way as homogeneous systems, because $\tilde{c} = \langle c^2\rangle^{1/2}$ is the averaged characteristic. Hence, in this case the entire set of equations can be expressed in terms of averaged variables.

Nonlinearity introduced by term $\langle V^2/c^2\rangle$ is most conveniently studied in media where the sound speed is pressure-independent. The role of nonlinearity was revealed by comparing self-simulating waves propagating over homogeneous and periodic media. The pressure profiles formed in the periodic two-component medium (k = 0.5; V_1/V_2 = 2) upon withdrawing a piston at a constant rate, are illustrated in Fig.1 as dimensionless dependences of pressure p/p_0 on the Lagrangian mass coordinate $\eta = s/\tau_0\ (p_0/\langle V_0\rangle)^{1/2}$ (τ_0 is the characteristic time). To correlate the results obtained for various media, the variables were normalized to the averaged specific volume and a small perturbation velocity. In the space of dimensionless variables this implies that at $p = p_0$ the pairs of quantities $\langle V\rangle$ and $\langle V^2/c^2\rangle$ for these media are identical. The results of numerical calculations are inaccurate because of the bend in the pressure profile at the rarefaction wave ends, for which reason they are inconsistent with their analytical counterparts.

In the limit of high pressures the slope of the curves tends to constant $\langle c^2\rangle^{-1/2}$. In the specific case of media with ξ-independent V/c^2, the profiles of self-simulating rarefaction wave will exhibit an extended straight segment, as they do in a homogeneous medium (curve *1* in Fig.1). Hence, in this respect periodic media are no different from homogeneous ones. In other periodic media the rarefaction wave profiles depart from the linear one. The parameters of the media were so selected that at $p \to \infty$, their averaged characteristics $\langle c^2\rangle$ were the same and curves *2* and *3* had identical slopes.

At moderate pressures, curves *2* and *3* diverge, which points to an appreciable effect of inhomogeneities.

So, the medium periodicity generally makes itself evident during wave propagation. The wave profiles carry much information about the wave structure. Following the wave evolution permits, to a certain extent, unraveling the wave structure [10]. However, it should be remembered that in the long wavelength model the structure period is infinitely small, and hence, the exact location of the structural elements on this interval (period) is unknown. For convenience, the $V = V(\xi)$ function is assumed decreasing, integrable, and one-valued. If the sound velocity is the same all over the periodic medium and constant, the reciprocal of the sought-for $\xi(V)$ function can be derived by applying the inverse Fourier transform [10]

$$\xi(V) = F^{-1}\left[\sum_{n=0}^{\infty}\frac{\langle V^{n+1}\rangle}{(n+1)!}\, i^n q^n\right](V) .$$

Coefficients $\langle V^n\rangle$ (n = 3,4,...) appearing in the above equation are easily estimated, provided the p-dependence of $\langle V^2\rangle$ (e.g., in the self-simulating rarefaction wave) is known. They are calculated by the recurrent relationship

$$\frac{d\langle V^{n+1}\rangle}{dp} = -(n+1)\frac{\langle V^{n+2}\rangle}{c^2}$$

which stem directly from the equation of state. As noted above, the mean value of $\langle V\rangle$ in the Euler frame of reference is inferred from the density of the medium. Thus, the medium structure is determined to within the specified accuracy.

Exact mathematical modeling of a native periodic medium discloses nonlinear effects associated with the medium structure, which show up during propagation of long high-amplitude waves and can serve as a diagnostic tool for unraveling the medium structure.

REFERENCES

1. G.M.Lyakhov, *Volny v Gruntakh i Poristykh Mnogokomponentnykh Sredakh* (Waves in Grounds and Porous Multi-Component Media), (Nauka, Moscow, 1982).
2. V.N.Rodionov, I.A.Sizov, and V.M.Tsvetkov, *Osnovy Geomekhaniki* (Fundamentals of Geomechanics), (Nedra, Moscow, 1986).
3. M.A.Sadovskii, L.G.Bolkhovitinov, and G.P.Pisarenko, *Deformiruemost' Geofizicheskoi Sredy i Seismicheskii Protsess* (Deformability of the Geophysical Medium and Seismic Process), (Nauka, Moscow, 1987).
4. V.A.Danilenko, *Doklady AN Ukrainy*, No.2, 87 (1992) (in Russian).
5. N.S.Bakhvalov and G.P.Panasenko, *Osrednenie Protsessov v Periodicheskikh Sredakh* (Averaging of Processes Occurring in Periodic Media), (Nauka, Moscow, 1984).
6. E.Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, (Springer, Heidelberg, 1980).
7. V.A.Vladimirov, V.A.Danilenko, and V.Yu.Korolevich, *Nelineinye Modeli Mnogokomponentnykh Relaksiruyushchikh Sred. Dinamika Volnovykh Struktur i Kachestvennyi Analiz. Preprint* (Nonlinear Models of Multi-Component Relaxing Media. Dynamics of Wave Structures and Qualitative Analysis. Preprint), (Institut Geofiziki, Kiev, 1990).
8. V.A.Vakhnenko, V.A.Danilenko, and V.V.Kulich, *Doklady AN USSR*, No.4, 93 (1991) (in Russian).
9. V.A.Vakhnenko, V.A.Danilenko, and V.V.Kulich, *Elementy Teorii Samoorganizatsii i Nelineinykh Volnovykh Protsessov v Prirodnykh Sredakh so Strukturoi. Preprint* (Elements of Self-Organization Theory and Nonlinear Wave Processes in Structured Native Media. Preprint), (Institut Geofiziki, Kiev, 1991).
10. V.A.Vakhnenko and V.V.Kulich, *Osrednennye Uravneniya Volnovoi Dinamiki Periodicheskoi Relaksiruyushchei Sredy. Preprint* (Averaged Equations of Wave Dynamics of a Periodic Relaxing Medium), (Institut Geofiziki, Kiev, 1991).