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Diagnostics of the medium structure by long wave of finite amplitude

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Abstract

The averaged systems of hydrodynamic equations for a structured medium in the Lagrangian and the Eulerian coordinates are discussed. In the general case, the equations cannot be reduced to the average hydrodynamic terms. Under propagation of long waves in media with structure, the non-linear effects appear and they are analyzed in the framework of the asymptotic averaged model. The heterogeneity in a medium structure always increases the non-linear effects for the long-wave perturbations. A new method for diagnostics of the properties of medium components by long non-linear waves is suggested (inverse problem). The mass contents of components in the media can be determined by this diagnostic method. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Traditionally, it was considered that in heterogeneous media with wavelength appreciably exceeding the size of the structural heterogeneities, the perturbations propagate in the same way as in homogeneous media [1–3]. In the framework of continuum mechanics [4,5] the known idealization of a real medium as homogeneous has enabled considerable success for describing the wave processes (see Ref. [3] and references therein). Recent experiments have shown that it is necessary to take into account the inner structure of a medium under propagation of the non-linear wave perturbations [6–9]. The information contained in the wave field evolution can be used as a tool when we want to establish the properties of the medium itself as well as to study the effects on various objects. The rigorous mathematical analysis by the method of asymptotic averaging [10,11] shows that the structure of the medium affects the non-linear wave processes even for perturbations with a wavelength λ that many times exceeds the size of the heterogeneity ε [11–14] of the medium. The heterogeneous medium can be modeled by a homogeneous one in an acoustic approach only [14]. In this work, using the previously suggested asymptotic averaged model of the heterogeneous media, we develop the further analysis of a weak non-linearity of the medium with the structure [14] under the condition of the propagation of long wavelength perturbations.

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The effect of the increase of non-linearity in the heterogeneous media in comparison with the homogeneous media for the long waves is found. It forms the basis of theoretical fundamentals of a diagnostic method to define the properties of heterogeneities in a medium by the long waves of finite amplitudes.

2. Asymptotic averaged model for structured medium

In this section we shall present briefly the asymptotic averaged model for structured medium developed in Refs. [12–14] in order to explain the effect of an increase of the non-linearity in the case of long waves propagating in such a medium.

The long non-linear waves have been investigated by using as an example a medium with regular structure. It is supposed that the microstructure elements of the medium ε are large enough that it is possible to submit to the laws of classic continuum mechanics. The analysis is based on the hydrodynamic approach. This restriction can be imposed for the modeling of non-linear waves in water-saturated soils, bubble media, aerosols, etc. The set of acceptable media could be extended to solid media where the powerful loads are studied in the condition that the strength and plasticity of the material can be neglected [15]. In the hydrodynamic approach we have considered the media without tangential stresses while there are equalities of the stresses as well as of mass velocities on boundaries of neighboring components. Also, we assume that the medium is barothropic. Individual components of the medium are considered to describe by the classical equations of hydrodynamics (Lagrangian mass coordinates)

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial m} = 0, \qquad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial m} = 0$$

and state equations

$$\mathrm{d}p = c^2\,\mathrm{d}\rho.$$

The notations are as generally accepted. Conditions for matching are the equality of the mass velocities and of pressures at boundaries of components. In contrast to the previous work [14], we have restricted ourselves to the consideration by the planar waves in a non-relaxing medium.

We study here a heterogeneous medium by a method of the asymptotic averaging [10,11] which is the combination of a multiscale method with a space averaging method. According to this method, the mass space coordinate m can be split into two independent coordinates: slow one, s, and fast one, ξ. Then

$$m = s + \varepsilon\xi, \quad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi}.$$

The slow coordinate s corresponds to a global variation of the wave field and it is a constant over the whole period of the structure of the medium while the fast one ξ traces variations of the field within the structure period. The dependent functions are presented in the form of power series over the structure period ε. For example,

$$p(m,t) = p^{(0)}(s,t,\xi) + \varepsilon p^{(1)}(s,t,\xi) + \varepsilon^2 p^{(2)}(s,t,\xi) + \cdots .$$

Functions $p^{(i)}$, $u^{(i)}$, $V^{(i)}$ are considered as the one-period functions over ξ. The period remains constant only in the Lagrangian mass coordinates that permit averaging. It was shown in Refs. [12–14] that in the zero approximation by ε/λ, pressure $p^{(0)}$ and mass velocity $u^{(0)}$ are constant within the period (i.e. they do not depend on ξ) but this is not correct for the specific volume $V^{(0)} = V^{(0)}(\xi)$. The independence of a variable on a fast coordinate ξ means that $u^{(0)} = \langle u^{(0)} \rangle$ and $p^{(0)} = \langle p^{(0)} \rangle$.

After integrating over the structure period of the equations containing the value of zero order of ε only, we obtain the averaged system of equations [12–14]

$$\frac{\partial \langle V^{(0)} \rangle}{\partial t} - \frac{\partial u^{(0)}}{\partial s} = 0, \quad \frac{\partial u^{(0)}}{\partial t} + \frac{\partial p^{(0)}}{\partial s} = 0, \tag{1}$$

$$\mathrm{d}\langle V^{(0)} \rangle = -\langle (V^{(0)})^2/c^2 \rangle\,\mathrm{d}p^{(0)}. \tag{2}$$

We define $\langle\,\cdot\,\rangle \equiv \int_0^1 (\cdot)\,\mathrm{d}\xi$ and then normalize such that $\int_0^1 \mathrm{d}\xi = 1$. We are restricted hereafter to zero approximations, and superscript (0) can be omitted. Choosing the wavelength λ large enough we can always reduce to zero the effect from other approximation terms.

The averaged system of Eqs. (1), (2) is integro-differential and in the general case is not reduced to the averaged variables p, u, $\langle V \rangle$. Consequently, the dynamic behavior of a medium cannot be modeled by means of a homogeneous medium even for long waves.

It is noted [14] that on the microlevel the action is statically uniform (waveless) because the pressure and the mass velocity do not depend on ξ. On this level the behavior of the medium adheres to the thermodynamic laws only. The macrolevel motion of the medium is described by the wave dynamic laws for averaged characteristics. On this hierarchic level Eqs. (1) and (2) do not change their form if, in an elementary cell, the components are broken and/or their location is changed. This means that Eqs. (1) and (2) define the motion of any quasi-periodic (statistically heterogeneous) medium. Thus, the asymptotic averaged model describes the long non-linear wave propagation in the structured medium.

In general, the individual components have a different compression under the propagation of the non-linear waves. The change of the averaged specific volume $\langle V \rangle$ differs from the change of specific volume for a homogeneous medium V_{hom} under the same loading. Consequently, the medium structure affects non-linear wave motion.

Introducing an effective average sound velocity by the formula

$$c_{\text{eff}} = \sqrt{\frac{\langle V \rangle^2}{\langle V^2/c^2 \rangle}}, \tag{3}$$

we obtain the traditional representation of the system of equations to describe the motion of the medium. It should be noted that c_{eff} is not an averaged characteristic, i.e. $c_{\text{eff}}^2 \neq \langle c^2 \rangle$. Evidently, the structure of the medium introduces a certain contribution to the non-linearity. In fact, if even $c \neq f(p)$, then in a general case the value c_{eff} is a function of pressure.

However, the pressure fields in the periodic and homogeneous media coincide within the acoustic waves only, if [13,14]

$$\langle V_0 \rangle = V_{0\,\text{hom}}, \quad \langle V_0^2/c_0^2 \rangle = (V_0^2/c_0^2)_{\text{hom}}. \tag{4}$$

For perturbations with the wavelength λ that many times exceeds sizes of the heterogeneity ε, the heterogeneous medium can be modeled by a homogeneous medium only in the acoustic approach.

To correlate the results obtained for various media, the variables are normalized to the averaged specific volume $\langle V \rangle$ and the initial sound velocity c_{eff}. It is seen that the acoustic waves in such media propagate in a similar way (see Eq. (4)).

We note that the initial equations of a motion (1) may be rewritten in the Eulerian system of coordinates. In Ref. [14] we obtained the following transformation between Lagrangian (s, t) and Eulerian (x, t_E) coordinates:

$$\mathrm{d}x = \langle V \rangle\,\mathrm{d}s + u\,\mathrm{d}t, \quad t_E = t. \tag{5}$$

Then Eq. (1) in the Eulerian coordinate system take the form (index E is omitted)

$$\frac{\partial \langle V \rangle^{-1}}{\partial t} + \frac{\partial u \langle V \rangle^{-1}}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \langle V \rangle \frac{\partial p}{\partial x} = 0. \tag{6}$$

It is convenient to define the fast Eulerian coordinate ζ as

$$\left(\frac{\partial \zeta}{\partial \xi}\right)_t = \frac{\tilde{\rho}}{\rho(\xi)}. \tag{7}$$

It should be noted that the average density $\tilde{\rho}$ in the Eulerian coordinates is a value usually used for density. A chain of identities

$$\langle V \rangle = \int_0^1 V(\xi)\,\mathrm{d}\xi = \int_0^1 V\frac{\rho(\xi)}{\tilde{\rho}}\,\mathrm{d}\zeta = \tilde{\rho}^{-1}$$

proves that $\langle V \rangle^{-1}$ is the mean density of the medium in the Eulerian coordinates. This value can be easily determined experimentally. At the same time the averaged values p and u coincide in both Lagrangian and Eulerian systems of coordinates.

Averaged equations of motion in the Lagrangian and Eulerian coordinates are analogous to equations for a homogeneous medium in corresponding coordinates. State equations differ essentially. A structure of the medium appears only in Eq. (2). The value $\langle V^2/c^2\rangle$ introduces an additional non-linearity.

3. Non-linear effects in structured medium

In this section we shall show that the medium structure always increases the non-linear effects under the propagation of long waves. A long wave with finite amplitude responds to the structure of the medium, and the non-linear effects in this medium increase in comparison with ones in the homogeneous medium. In addition to the previous analysis of the sound velocity in homogeneous and heterogeneous media (see Eq. (14) in Ref. [14]), we consider now an evolution equation with non-linear term and compare the coefficients of non-linearity in these media.

Let us obtain an evolution equation that takes into account a weak non-linearity. First of all, we have to note that mass velocity u is related to the pressure p by means of [13]

$$u = \int_{p_0}^{p} \sqrt{\langle V^2/c^2\rangle}\,\mathrm{d}p. \tag{8}$$

A functional dependence of an average specific value on the pressure increment $p' = p - p_0$ with an accuracy $O(p'^2)$ can be presented as a series

$$\langle V\rangle(p) = \langle V\rangle_0 + \left.\frac{\mathrm{d}\langle V\rangle}{\mathrm{d}p}\right|_{p=p_0} p' + \frac{1}{2}\left.\frac{\mathrm{d}^2\langle V\rangle}{\mathrm{d}p^2}\right|_{p=p_0} p'^2.$$

In this case the system of Eq. (6) can be written as

$$\langle V\rangle_0\frac{\partial u}{\partial x} + \left\langle\frac{V^2}{c^2}\right\rangle_0\frac{\partial p'}{\partial t} - \frac{1}{2}\left.\frac{\mathrm{d}^2\langle V\rangle}{\mathrm{d}p^2}\right|_{p=p_0}\frac{\partial p'^2}{\partial t} = 0,$$

$$\frac{\partial u}{\partial t} + \langle V\rangle_0\frac{\partial p'}{\partial x} = 0.$$

Hereinafter index 0 is omitted. The relationship $u(\partial p'/\partial x) = p'(\partial u/\partial x)$ follows from Eq. (8) with an assumed accuracy $O(p'^2)$, and it was used for derivation of the first equation. An evolution equation for one variable assumes the form

$$\langle V\rangle^2\frac{\partial^2 p'}{\partial x^2} - \left\langle\frac{V^2}{c^2}\right\rangle\frac{\partial^2 p'}{\partial t^2} + \frac{1}{2}\frac{\mathrm{d}^2\langle V\rangle}{\mathrm{d}p^2}\frac{\partial^2 p'^2}{\partial t^2} = 0. \tag{9}$$

Now let us consider the waves propagating in one direction, then with indicated accuracy we can write $(\sqrt{\langle V^2/c^2\rangle}/\langle V\rangle)\partial/\partial t + \partial/\partial x \to 2\partial/\partial x$ (see, for example, Section 93 in Ref. [5]). Thus, after factorization of Eq. (9) we get

$$\left(\frac{\sqrt{\langle V^2/c^2\rangle}}{\langle V\rangle}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p' - \frac{1}{2}\left\langle\frac{V^2}{c^2}\right\rangle^{-1}\frac{\mathrm{d}^2\langle V\rangle}{\mathrm{d}p^2}p'\frac{\partial p'}{\partial x} = 0.$$

A coefficient of non-linearity β_p for the structured medium when the sound velocities in the individual components are independent of the pressure $c \neq f(p)$, can be presented as

$$\beta_p = \frac{1}{2}\langle V\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2}\frac{\mathrm{d}^2\langle V\rangle}{\mathrm{d}p^2} = \frac{\mathrm{d}(u + c_{\mathrm{eff}})}{\mathrm{d}p} = \langle V\rangle\left\langle\frac{V^3}{c^4}\right\rangle\left\langle\frac{V^2}{c^2}\right\rangle^{-3/2}.$$

For all cases $\beta_p > 0$. For a homogeneous medium with $\mathrm{d}c/\mathrm{d}p = 0$ we have $\beta_{p\,\mathrm{hom}} = V/c$.

In certain media the value V/c^2 does not change within the period. Individual elements of the structure respond to the pressure variations so that a relative structure does not change, i.e. a ratio $V(\xi,p)/V(\xi,p_0)$ does not depend on ξ. In this case, the value $c_{\mathrm{eff}} = \sqrt{\langle c^2\rangle}$ derived from Eq. (3), is an averaged characteristic. Consequently, the system of equations may be presented in the averaged variables p, u, $\langle V\rangle$, $c_{\mathrm{eff}} = \sqrt{\langle c^2\rangle}$. Heterogeneity does not introduce additional non-linearity for these media. Such media behave like homogeneous media under the action of non-linear wave perturbations.

For media when the sound velocity is independent of the pressure ($c \neq f(p)$) it is possible to show that a heterogeneity of the medium, in the general case, introduces additional non-linearity. Let us consider the ratio of the non-linearity coefficients for heterogeneous and homogeneous media. In the space of dimensionless normalized variables this implies that at $p = p_0$ we have $\langle V\rangle_0 = 1$ as well as $\langle V^2/c^2\rangle_0 = 1$ for compared media.

Using conditions (3) and (4) we can obtain

$$\frac{\beta_p}{\beta_{p\,\mathrm{hom}}} = \langle V \rangle \left\langle \frac{V^3}{c^4} \right\rangle \left\langle \frac{V^2}{c^2} \right\rangle^{-2} \geqslant 1. \tag{10}$$

This inequality is the well-known Cauchy–Schwarz inequality (formula 15.2–3 in Ref. [16]). By designating $a^2 \equiv V/c^2, b^2 \equiv V^2/c^2$, it is easy to notice that the equals sign is realized if and only if $a^2 = \mathrm{const.}$ (see Sections 14.2–6 in Ref. [16]), i.e. when V/c^2 does not vary within the period. This heterogeneous medium has been considered above. For all other heterogeneous media for which the value V/c^2 changes within period, the inequality is realized in Eq. (10). So, in a heterogeneous medium β_p is always greater than $\beta_{p\,\mathrm{hom}}$ in a homogeneous medium. Thus, it is proved that, in the general case, the heterogeneity in a medium structure introduces additional non-linearity. This effect provided the basis for a new method of diagnostics to define the properties of multicomponent media using the propagation of long non-linear waves in such media.

4. Fundamentals of new diagnostic method

The structure of the medium affects the wave field. There are different methods which allow detection of gas bubbles and/or cracks in liquid [17], concrete [18], and ice covers [19] by means of the non-linear effects.

In the following part, we describe our new diagnostic method for the properties of a medium. The features of the motion of finite amplitude long waves and the effect of the increase of non-linearity in the heterogeneous medium in comparison with homogeneous medium form the basis for the development of the theoretical fundamentals of the diagnostic method. In this method the properties of medium heterogeneities are defined by long waves of finite amplitudes. We shall have shown that the dependence V/c^2 on fast Eulerian coordinate ζ (see Eq. (7)) is defined.

It should be kept in mind that the period of the medium structure is infinitely small in the long-wave model, so it is not always possible to indicate reliably the location of the structure elements inside the period. Then, for definiteness, the $V/c^2 = V/c^2(\zeta)$ function is assumed decreasing, integrable, mutually one-valued function on the section $\zeta \in [0, 1]$ and is equal to zero outside it. Thus, the wave evolution allows with an inherent accuracy to define the structure of the medium. In the final result, the mass contents of the particular components can be denoted using this method.

It is known from theory of probability that a distribution function $f(x)$ (any one-valued, integrable, positive function) can be expressed by its central moments

$$\alpha_n = \int_{-\infty}^{\infty} x^n f(x)\,\mathrm{d}x$$

by means of the inverse Fourier transform

$$f(x) = F^{-1}\left[\sum_{n=0}^{\infty} \alpha_n \mathrm{i}^n \frac{q^n}{n!}\right]$$

if series $\sum_{n=0}^{\infty} |\alpha_n|(s^n/n!)$ converges absolutely for some $s > 0$ (see Section 18.3.7 in Ref. [16]). Let us consider a chain of transformations using Eq. (7)

$$\begin{aligned}\langle V(V/c^2)^n \rangle &= \int_0^1 V(\xi)\left(\frac{V}{c^2}\right)^n \mathrm{d}\xi = \int_0^1 V\left(\frac{V}{c^2}\right)^n \frac{\rho}{\tilde{\rho}}\,\mathrm{d}\zeta \\ &= \langle V \rangle \int_{-\infty}^{\infty} \left(\frac{V}{c^2}\right)^n \frac{\mathrm{d}\zeta}{\mathrm{d}V/c^2}\,\mathrm{d}V/c^2 \\ &= n\langle V \rangle \int_{-\infty}^{\infty} \left(\frac{V}{c^2}\right)^{n-1} \zeta\,\mathrm{d}V/c^2 \\ &= n\langle V \rangle \alpha_{n-1}.\end{aligned}$$

A central moment α_{n-1} of the function $\zeta = \zeta(V/c^2)$ is expressed by means of $\langle V(V/c^2)^n \rangle$. Finally, we find the inverse function

$$\zeta = F^{-1}\left[\sum_{n=0}^{\infty} \frac{\langle V(Vc^{-2})^{n+1} \rangle}{(n+1)!\langle V \rangle}\,\mathrm{i}^n q^n\right] \tag{11}$$

as desired. The physical value Vc^{-2} is bounded by some constant M, hence

$$\alpha_n = \int_{-\infty}^{\infty} (V/c^2)^n \zeta\,\mathrm{d}V/c^2 \leqslant \int_0^M (V/c^2)^n\,\mathrm{d}V/c^2 = \frac{M^{n+1}}{n+1}.$$

The series $\sum_{n=0}^{\infty}(|\alpha_n|s^n)/n! \leqslant \sum_{n=0}^{\infty} M^{n+1}s^n/(n+1)!$ converge at $s < M^{-1}$. Consequently, the power series (11) also converges.

The coefficients $\langle V(Vc^{-2})^n\rangle$ in Eq. (11) can be easily calculated if we know the functional dependence $\langle V\rangle(p)$ or $\langle V^2/c^2\rangle(p)$. They can be successively defined by the recurrence relation

$$\frac{\mathrm{d}\langle V(Vc^{-2})^n\rangle}{\mathrm{d}p} = -(n+1)\langle V(Vc^{-2})^{n+1}\rangle \tag{12}$$

that follows directly from the state equation. With mentioned accuracy it is possible to diagnose the structural properties of the medium.

Application of the suggested method is connected with the finding of the coefficients $\langle V(Vc^{-2})^n\rangle$ for power series (11). These coefficients can be obtained from the features of wave field evolution. The advantages of diagnostics by means of wave fields is evident, especially for the media with complex structures, in particular for natural media.

A possible way to obtain the functional dependence of $\langle V\rangle$ on p is to perform an experiment to define the parameters of shock waves. The shock wave velocity in Lagrangian mass coordinates $D = \mathrm{d}s/\mathrm{d}t$ (dimension $[D]$, kg/s) and/or mass velocity u as well as pressure p after shock wave front can be experimentally measured. The value $\langle V\rangle$ is calculated from the relationships on the shock front

$$D = \sqrt{(p-p_0)/(\langle V_0\rangle - \langle V\rangle)},$$

$$u - u_0 = \sqrt{(p-p_0)(\langle V_0\rangle - \langle V\rangle)}.$$

These relationships follow from averaged motion equations [14]. After the measurement of the shock wave parameters for various pressures p, we can obtain the dependence $\langle V\rangle = \langle V\rangle(p)$. Then the recurrence formula (12) is applied to obtain $\langle V(Vc^{-2})^n\rangle$ at $n \geqslant 1$ for Eq. (11).

The self-similar rarefaction wave [12] can be considered as a universal instrument to define the coefficients $\langle V(Vc^{-2})^n\rangle$. Self-similar motion of the rarefaction wave, as appears from relationship (2.5) of Ref. [13] $\mathrm{d}s/\mathrm{d}t = \langle V^2/c^2\rangle^{-1/2}$, gives the propagation velocity $\mathrm{d}s/\mathrm{d}t$ of separate parts of the wave profile under various pressures. The evolution of the profile of the rarefaction wave makes it possible to define the dependence $\langle V^2/c^2\rangle = \langle V^2/c^2\rangle(p)$, and, consequently, the values $\langle V(Vc^{-2})^n\rangle$ at $n \geqslant 2$ which can be found from Eq. (12).

We pointed out a few ways by means of which, from our point of view, it is possible to find the required dependencies from experiments. Certain difficulties for the application of this method can be connected with the following. The experimental data are always defined with some accuracy, and the application of Eq. (12) will lead to the increase of the magnitude of error for high-order derivatives. This requires that a limited number of the terms should be used in series (11). Consequently, it is necessary to study the accuracy of the reconstruction of medium structure in the case when we know only several first terms in series (11).

5. Approximation of diagnosed medium by layer medium

Diagnostics of the structured medium properties by long non-linear waves is connected with the definition of values $\langle V(Vc^{-2})^n\rangle$. As indicated above, there is a problem related to the accuracy of the description of the structure by finite series (11).

Now, we shall have shown that the partial sum of series (11) is a step-function and approximates the desired function $\zeta = \zeta(V/c^2)$ with certain accuracy, namely the diagnosed medium can be approximated by a layer medium.

Let us write down the chain of the identities for any integrable function

$$\begin{aligned}2\pi f(-x) &= F[F[f(x)](q)](x)\\ &= F\left[\sum_{n=0}^{\infty}\frac{\mathrm{i}^n q^n}{n!}\alpha_n\right]\\ &= \sum_{n=0}^{\infty}\frac{\mathrm{i}^n\alpha_n}{n!}2\pi(-\mathrm{i})^n\delta^{(n)}(x).\end{aligned}$$

Here we used the known relationships for the Fourier transform [16]. Hence, any integrable function can be represented by a series

$$f(-x) = \sum_{n=0}^{\infty}\frac{\alpha_n}{n!}\delta^{(n)}(x). \tag{13}$$

Consider the step-function $f_1(x)$ consisting of N steps:

$$f_1(x) = \begin{cases} \varphi_1, & 0 < x \leqslant b_1, \\ \varphi_2, & b_1 < x \leqslant b_2, \\ \vdots & \vdots \\ \varphi_N, & b_{N-1} < x \leqslant b_N, \end{cases}$$

where the desired function $f(x)$ will be approximated. Evidently, by increasing the number of steps N and choosing the values φ_i, b_i, any integrable function $f(x)$ can be approximated by the step-function $f_1(x)$. It is convenient to use a notation

$$f_1(-x) = \varphi_1[\Theta(x+b_1) - \Theta(x)] + \varphi_2[\Theta(x+b_2) - \Theta(x+b_1)] + \cdots + \varphi_N[\Theta(x+b_N) - \Theta(x+b_{N-1})]. \quad (14)$$

The Heavyside function $\Theta(x+b)$ can be expanded into a Taylor series in a neighborhood of point x

$$\Theta(x+b) = \Theta(x) + \sum_{n=1}^{\infty} \frac{b^n}{n!}\Theta^{(n)}(x).$$

We equate functions (13) and (14), and consider that the number of steps for function $f_1(x)$ is infinitely larger, and in this case we obtain

$$\varphi_1 \sum_{n=0}^{\infty} \frac{b_1^{n+1}}{(n+1)!}\delta^{(n)}(x) + \varphi_2 \sum_{n=0}^{\infty} \frac{b_2^{n+1} - b_1^{n+1}}{(n+1)!}\delta^{(n)}(x) + \cdots + \varphi_N \sum_{n=0}^{\infty} \frac{b_N^{n+1} - b_{N-1}^{n+1}}{(n+1)!}\delta^{(n)}(x) + \cdots = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!}\delta^{(n)}(x). \quad (15)$$

This relationship shows that when we use the partial sum of series on the right-hand side of Eq. (15) $\sum_{n=0}^{2N-1}(\alpha_n/n!)\delta^{(n)}(x)$ and also N leading terms on the left-hand side, then the desired function $f(x)$ is approximated by the step-function $f_1(x)$ with N steps. In other words, if it is necessary to restore the medium structure by means of N periodic repeated layers, then $2N-1$ moments α_n, i.e. the values $\langle V(Vc^{-2})^n \rangle$ should be known.

For convenience, we write down relation (15) in the expanded form. For this purpose, we multiply it by x^n and integrate over x. We obtain the nonlinear system of the equations in the unknowns $b_1, b_2, \ldots, b_N, \varphi_2, \varphi_3, \ldots, \varphi_N$ (variable $\varphi_1 = 1$ owing to normalization)

$$\begin{aligned} &\varphi_1 b_1 + \varphi_2(b_2 - b_1) + \varphi_3(b_3 - b_2) \\ &\quad + \cdots + \varphi_N(b_N - b_{N-1}) = \alpha_0, \\ &\varphi_1 b_1^2 + \varphi_2(b_2^2 - b_1^2) + \varphi_3(b_3^2 - b_2^2) \\ &\quad + \cdots + \varphi_N(b_N^2 - b_{N-1}^2) = 2\alpha_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\varphi_1 b_1^{2N-1} + \varphi_2(b_2^{2N-1} - b_1^{2N-1}) \\ &\quad + \varphi_3(b_3^{2N-1} - b_2^{2N-1}) \\ &\quad + \cdots + \varphi_N(b_N^{2N-1} - b_{N-1}^{2N-1}) \\ &\quad = (2N-1)\alpha_{2N-2}. \end{aligned} \quad (16)$$

Now, if b_i implies the partition of $(V/c^2)_i$, and φ_i implies the partition of ζ_i, we can obtain the system of Eqs. (16) to define the medium structure. Solution of these equations gives the information about the component properties of the medium, namely, the value V/c^2 on the structure period $\zeta \in [0,1]$ is found in the form of the step-function.

Let us note the special case of a periodic medium for which the value V/c^2 is constant within the period. This medium, as we already know, does not differ from a homogeneous one for the propagation of the long non-linear waves. The same result follows from a system (16). Indeed, for homogeneous media the moments α_n are equal to $\alpha_n = \langle V(Vc^{-2})^{n+1} \rangle / ((n+1)\langle V \rangle) = b^{n+1}/(n+1)$. Here, the conditions of normalization $\langle V^2/c^2 \rangle_0 = (V^2/c^2)_0 = 1$, $\langle V \rangle_0 = V_0 = 1$ have been used as before. Therefore, the values in the right-hand side of Eq. (16) are equal to the $b \equiv Vc^{-2} = \text{const}$. It is easy to see that the solution of system is $b_1 = b_2 = \cdots = b_N = b$, $\varphi_1 = 1$ (where φ_i is any value for $i \geqslant 2$). This corresponds to the layer medium, for which $V/c^2 \neq f(\zeta)$; in particular, this medium can be a homogeneous one.

According to the asymptotic averaged model of a structured medium the period of the structure is infinitely small, and this diagnostic method cannot give the exact location of the structure elements inside the period. Hence, using this method, only the mass contents of the particular components can be determined.

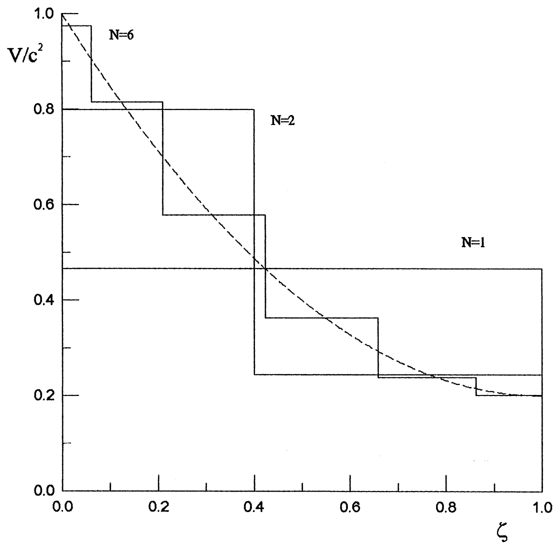

Fig. 1. Approximation of the diagnosed medium (dotted curve) by N-component media.

We present as an example the results of the calculation to define the structure of layer media which can properly approximate the diagnosed medium for the case of $V/c^2 = 0.2 + 0.8(1-\zeta)^2$ (see Fig. 1). In order to approximate the diagnosed medium by layer periodic medium, which has N layers within the period, it is necessary to know $2N-1$ values $\langle V(Vc^{-2})^n \rangle$ for finite series (11). If we regard that the $2N-1$ averaged characteristics $\langle V(Vc^{-2})^n \rangle$ coincide for the diagnosed medium and for the layer medium, these averaged values at $n \leqslant 2N-1$ can be calculated from known distribution $V/c^2 = 0.2 + 0.8(1-\zeta)^2$ within the period. At $n > 2N-1$ the values $\langle V(Vc^{-2})^n \rangle$ for diagnosed medium and for approximated layer medium are different. The distributions of V/c^2 within the period for diagnosed medium (dotted curve) and for approximated media with N components are shown in Fig. 1. So, we have illustrated the accuracy of the approximation of the diagnosed medium by the finite series (11).

Thus, the new method for the diagnostics of the medium characteristics by long non-linear waves is suggested on the basis of the asymptotic averaged model of structured medium.

6. Conclusions

In conclusion, we have discussed the averaged systems of hydrodynamic equations in the Lagrangian and Eulerian coordinates. These systems are not expressed in the average hydrodynamic terms $p, u, \langle V \rangle$ and, consequently, the dynamic behavior of a medium cannot be modeled by means of a homogeneous medium even for long waves, if they are non-linear. The structure of the medium affects the non-linear wave propagations. The heterogeneity in a medium structure always introduces additional nonlinearity in comparison with homogeneous medium. This effect enables us to form the theoretical fundamentals of the new diagnostic method to define the characteristics of a heterogeneous medium using the long waves of finite amplitudes (inverse problem). The mass contents of the particular components can be denoted by this diagnostic method.

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