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An asymptotic averaged model of non-linear long waves propagation in media with a regular structure

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Abstract

An asymptotic averaged model describing long non-linear wave propagation in non-equilibrium structured media is suggested. The averaged system of the hydrodynamic equations is an integro-differential system which cannot be reduced to the system of averaged hydrodynamic values. It is proved that the long wave with finite amplitude responds to the structure of the medium so that the structured medium cannot be modeled as a homogeneous medium. However, for long acoustic waves the internal structure of a medium manifests itself only by means of the dispersive dissipative properties, and the dynamic behavior of the medium can be described in the framework of a homogeneous relaxing medium. The important result of this model is that the structure of a medium always increases the non-linear effects under the propagation of long waves, and that non-linearity takes place even for media with the components described by the linear law. On a microstructural level of a medium, the dynamic behavior adheres only to the thermodynamic laws, wherein the change of the structure eventually affects the macro wave motion. The model justifies the one-velocity continuous models. As an example, a comparison of the suggested model with the known Lyakhov's model for natural multicomponent media is carried out.

Keywords: Non-linear wave; Structured medium; Asymptotic model; Dynamic state equation

1. Introduction

Natural media are not structureless. Recent experiments have shown that the internal structure of a medium affects wave motions [1–7]. The presence of non-homogeneity complicates the problem and, at the same time, is manifested fully under the propagation of non-linear waves. High-gradient fast processes, such as earthquakes, explosions, etc., lead to non-reversible processes [1, 2]. The principal parts of the problem are phenomena related to the non-linear behavior of natural media such as (a) soliton-like properties of *P*-waves [8], and (b) a larger increase of non-linear effects in structured media compared to homogeneous ones [3–7].

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Models with the different degrees of complexity have been used to describe the wave processes in heterogeneous media. Traditionally, in conditions of local equilibrium, media are modeled without considering their structure. In the framework of continuum mechanics the known idealization of a real medium as homogeneous has enabled considerable success for describing wave processes (see, for instance, [9–11]). The continuum models are commonly applied to the mixtures in which the dispersive dissipative properties take into account the interactions between the components [12–16]. On this level the media are modeled in the framework of a homogeneous elastic, viscous elastic and elastic plastic media [13, 17]. In this case the features of a medium's structure are taken into account indirectly through the kinetic parameters (relaxation time, the viscous coefficients etc.) [5, 6, 9, 12–17].

The model of the multivelocity interpenetratable continua was developed to describe the dynamic behavior of multicomponent media by the methods of classical continuum mechanics [18] and statistical physics [19]. A fundamental assumption in the theory of mixtures [12] coincides with an assumption in the model of the multivelocity interpenetratable continua [18]; namely, that each micro-volume dv is occupied by a particle belonging to each constituent. The equations of motion written for each component involve the terms describing the mass, force and energy interactions between the components. The problem is complicated by the necessity to attract, in the general case, the experimental data in order to establish the theoretical relations between macroparameters on the level of the components interaction. Further, it has been possible to define the components interaction, then these models would have been indispensable to the theory of multicomponent media. Review and application of the different models can be found in the monograph by Rajagopal and Tao [12].

In all the above models, the formalism of continuum mechanics which was used is based on the principle of local action as well as on the generalization of the mechanics laws relating the point mass to the continuum [10].

When going from integral equations to differential balance equations, the existence of a differentially small microvolume dv is assumed. While on the one hand, this volume is so small that the mechanics laws of the point mass have been expanded to the whole microvolume, on the other, the volume contains so many structural elements of the medium that, in this sense, it can be regarded as macroscopic in spite of its smallness as compared to the entire volume occupied by the medium. So, the transition to the differential balance equations is based on the assumption of the smallness of the microstructural scales ε in comparison with the characteristic macroscopic scale of the flow λ, and the transition should be made to the limit $\varepsilon/\lambda \to 0$. Contraction of the volume dv to the point, in the general case, is correct for the continuous functions [10, 12]. This means that all points inside the differentially small volume are equivalent. Hence, if the mixture is considered, the equivalence of the points means that the field characteristics averaged over dv should be used. Consequently, it is assumed that the equations of motion can be written using the average terms such as density, mass velocity and pressure which are ascribed to each component separately. We note that in these models the sizes of components are not explicitly included.

The application of models of a homogeneous medium to the description of the dynamic wave processes in a structured natural medium encounters certain principal difficulties [1, 2, 6, 17]. We will take into account the structure of a medium on a macrolevel. We abandon the assumption that the differentially small volume dv contains all components of the medium, although we will consider the long wave approach when wavelength λ is much larger than the characteristic length of the structure of a medium ε. We consider a structured medium (Fig. 1) in which separated components are considered as a homogeneous medium (the differentially small volume dv is much smaller than the characteristic size of a particular component). We use an asymptotic averaged model for the description of wave processes in non-equilibrium heterogeneous media [20, 21]. In this case the integral differential system of equations that has been obtained cannot be reduced to the average terms (pressure, mass velocity, specific volume), and it contains the terms with the characteristic size of particular components.

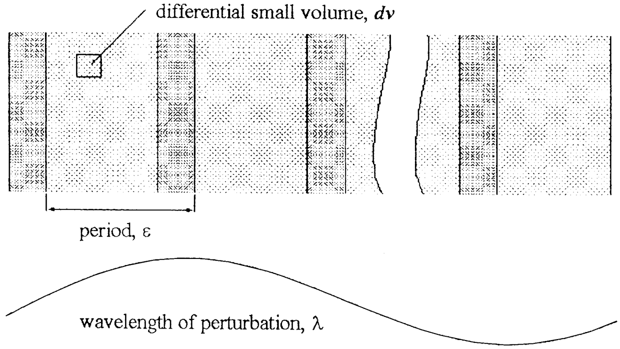

Fig. 1. Model medium with two homogeneous components in the period.

The suggested model justifies the one-velocity continuous model; in particular, we have included a comparison with the known Lyakhov's model for natural multicomponent media [16, 22]. A rigorous mathematical proof is provided to show that on the acoustic level, the only sufficient characteristics for describing the propagation of long waves are the dispersive dissipative properties of the medium, and in this case, the dynamic behaviour of the medium can be modeled by a homogeneous relaxing medium. However, the long wave within finite amplitude responds to the structure of the medium so that the behavior of the structured medium cannot be properly modeled by a homogeneous medium. The important result foretold by this model is that, for a wave with a finite amplitude the medium's structure (in particular the presence of microcracks) leads to non-linear effects, even if the separated components of the medium are described by a linear law.

2. Asymptotic averaged model

The simplest case of heterogeneous media for which the effect of the structure can be analyzed are media with a regular structure. Features of the propagation of long wave perturbations has been investigated by using as an example, a periodic medium under conditions of an equality of stresses and mass velocities on the boundaries of neighboring components. It is supposed that the microstructure elements of medium ε (see Fig. 1) are large enough that it is possible to submit to the laws of classical continuum mechanics. We also suppose that a medium is barothropic. We consider that the properties of the medium, such as density, sound velocity and relaxation time vary in a periodic manner (although this assumption is unessential in the final result). We have used the hydrodynamics approach and considered the media without tangential stresses. This restriction is imposed for the modeling of powerful loads as well as non-linear waves in water-saturated soils, bubble media, etc. [16].

In the Lagrangian coordinate system (l, t) the equations of one-dimensional motion for each element of the regular structure medium have the form

$$\frac{\partial r^{\nu}}{\partial l^{\nu}} = \frac{V}{V_0}, \quad u = \frac{\partial r}{\partial t}, \quad \frac{\partial u}{\partial t} + V_0\left(\frac{r}{l}\right)^{\nu-1}\frac{\partial p}{\partial l} = 0. \quad (1)$$

The equation of continuity can also be used in the alternative form

$$\frac{\partial V}{\partial t} - \nu V_0 \frac{\partial r^{\nu-1} u}{\partial l^{\nu}} = 0.$$

Here $V = \rho^{-1}$ is the specific volume, ν is a parameter of symmetry, where $\nu = 1$ – planar, $\nu = 2$ – cylindrical, $\nu = 3$ – spherical; the index 0 relates to the initial state. The other notations are those that are generally *accepted.*

Conditions for matching are the equality of the mass velocities and pressures on the boundaries of the components.

Considering the models of a relaxing medium as more general than the equilibrium models for describing the evolution of high-gradient waves, we will take into account the relaxing processes within each component. Thermodynamic equilibrium is disturbed by the propagation of fast perturbations in the medium. There are processes of the interaction that tend to return the medium to equilibrium. In essence, the change of macroparameters (pressure, mass velocity, density) caused by this interaction is a relaxation process. There are two limiting cases:

(a) the lack of relaxation (inner interaction processes are frozen), and the "frozen" sound velocity is defined as $c_f^2 = \mathrm{d}p_f/\mathrm{d}\rho$;
(b) the complete relaxation (there is the local thermodynamic equilibrium); the equilibrium sound velocity takes the form $c_e^2 = \mathrm{d}p_e/\mathrm{d}\rho$.

Slow and fast processes are compared with respect to each other by means of the relaxation time τ. The dynamic equation of state is applied to account for the relaxation effects

$$\mathrm{d}\rho = c_\mathrm{f}^{-2}\mathrm{d}p - \tau^{-1}(\rho - \rho_\mathrm{e})\,\mathrm{d}t. \tag{2}$$

The equilibrium equations of state are considered to be known $\rho_\mathrm{e} - \rho_0 = \int_{p_0}^{p} c_\mathrm{e}^{-2}\,\mathrm{d}p$.

The substantiation of Eq. (2) within the framework of the thermodynamics of irreversible processes has been given in [23–26]. To our knowledge the first work in this field was the article by Mandelshtam and Leontovich (see Section 81 in [24]). We note that the mechanisms of the exchange processes are not defined concretely when deriving Eq. (2), and the thermodynamic and kinetic parameters appear only in this equation. These characteristics can be found experimentally.

The phenomenological approach for describing the relaxation processes in hydrodynamics has been developed in many works [13, 16, 24, 25]. The dynamic equation of state was used (a) for describing the propagation of sound waves in a relaxing medium [24], (b) for taking into account the exchange processes within media (gas–solid particles) [25], (c) for studying wave fields in gas–liquid media [13] and in soil [16]. In most works, the equation of state has been derived from the concept of concrete mechanism for the inner process. Within the context of mixture theory, Biot [14] attempted to account for the non-equilibrium in velocities between components directly in the equations of motion in the form of dissipative terms.

We assume that the relaxation time and sound velocities do not depend on time, but that they are functions of pressure and the individual properties of the components. This means that in the process of a relaxation interaction we can take into account the exchange of moment and heat but not that of mass. Peculiarities of the intrastructure interaction are determined by the dynamic equation of state for each component.

A regularity of structure and a non-linearity of long-wave processes investigated here specify the choice of mathematical methods. One way of studying this heterogeneous medium is by the method of asymptotic averaging of equations with high-oscillating coefficients [27, 28]. The essence of this method is the application of a multiscale method in combination with space averaging. In accordance with this method, the mass space coordinate $m = l^\nu/V_0$ is divided into two independent coordinates: slow – s and fast – ξ, wherein

$$m = s + \varepsilon\xi, \quad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{3}$$

The slow coordinate, s, corresponds to a global change of the wave field and is a constant during a period, while the fast one, ξ, traces variations of a field in the structure period. The dependent functions are presented as a degree series over the structure period ε. For example,

$$\begin{aligned} p(m,t) &= p^{(0)}(s,t,\xi) + \varepsilon p^{(1)}(s,t,\xi) \\ &\quad + \varepsilon^2 p^{(2)}(s,t,\xi) + \cdots, \\ r^\nu(m,t) &= (r^\nu)^{(0)}(s,t,\xi) + \varepsilon (r^\nu)^{(1)}(s,t,\xi) \\ &\quad + \varepsilon^2 (r^\nu)^{(2)}(s,t,\xi) + \cdots, \end{aligned} \tag{4}$$

where $p^{(i)}$, $u^{(i)}$, $V^{(i)}$, $r^{(i)}$ are defined as the one-period functions of ξ. In the Lagrangian mass coordinates the period is a constant which allows the averaging procedure to be performed.

It may be proved that $p^{(0)} = p^{(0)}(s,t)$, $p^{(1)} = p^{(1)}(s,t)$, $u^{(0)} = u^{(0)}(s,t)$, $(r^\nu)^{(0)} = (r^\nu)^{(0)}(s,t)$ are independent of the fast variable ξ. Indeed, after substitution of Eqs. (3) and (4) into the initial equations of motion, we obtain

$$\varepsilon^{-1}\frac{\partial(r^{\nu})^{(0)}}{\partial\xi}+\varepsilon^{0}\left(\frac{\partial(r^{\nu})^{(0)}}{\partial s}+\frac{\partial(r^{\nu})^{(1)}}{\partial\xi}-V^{(0)}\right)+\cdots=0,$$

$$\varepsilon^{0}\left(u^{(0)}-\frac{\partial r^{(0)}}{\partial t}\right)+\cdots=0,$$

$$\varepsilon^{-1}\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(0)}}{\partial\xi}+\varepsilon^{0}\left(\frac{\partial u^{(0)}}{\partial t}+\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(0)}}{\partial s}+\nu(r^{\nu-1})^{(1)}\frac{\partial p^{(0)}}{\partial\xi}+\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(1)}}{\partial\xi}\right)+\cdots=0,$$

$$-\varepsilon^{-1}\nu\frac{\partial(r^{\nu-1})^{(0)}u^{(0)}}{\partial\xi}+\varepsilon^{0}\left(\frac{\partial V^{(0)}}{\partial t}-\nu\frac{\partial(r^{\nu-1})^{(0)}u^{(0)}}{\partial s}-\nu\frac{\partial(r^{\nu-1})^{(1)}u^{(0)}}{\partial\xi}-\nu\frac{\partial(r^{\nu-1})^{(0)}u^{(1)}}{\partial\xi}\right)+\cdots=0.$$

According to the general theory of the asymptotic method, the terms of equal powers of ε should vanish independently of each other. Thus, $\partial p^{(0)}/\partial\xi=0$, $\partial u^{(0)}/\partial\xi=0$, $\partial(r^{\nu})^{(0)}/\partial\xi=0$, i.e. $p^{(0)}=p^{(0)}(s,t)$, $u^{(0)}=u^{(0)}(s,t)$, $r^{(0)}=r^{(0)}(s,t)$ are independent of ξ. Furthermore

$$\frac{\partial(r^{\nu})^{(0)}}{\partial s}+\frac{\partial(r^{\nu})^{(1)}}{\partial\xi}=V^{(0)},\qquad u^{(0)}=\frac{\partial r^{(0)}}{\partial t},$$

$$\frac{\partial u^{(0)}}{\partial t}+\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(0)}}{\partial s}+\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(1)}}{\partial\xi}=0,$$

$$\frac{\partial V^{(0)}}{\partial t}-\nu\frac{\partial(r^{\nu-1})^{(0)}u^{(0)}}{\partial s}-\nu u^{(0)}\frac{\partial(r^{\nu-1})^{(1)}}{\partial\xi}-\nu(r^{\nu-1})^{(0)}\frac{\partial u^{(1)}}{\partial\xi}=0. \tag{5}$$

Thus, we can average the equations during the period ξ. We define, $\langle\cdot\rangle=\int_0^1(\cdot)\,\mathrm{d}\xi$, and perform the normalization $\int_0^1\mathrm{d}\xi=1$. Since $p^{(1)}$, $u^{(1)}$ and $r^{(1)}$ are periodic, integrals, such that $\langle\partial p^{(1)}/\partial\xi\rangle=0$, $\langle\partial u^{(1)}/\partial\xi\rangle=0$, $\langle\partial r^{(1)}/\partial\xi\rangle=0$. Moreover, as $\langle u^{(0)}\rangle=u^{(0)}$, $\langle p^{(0)}\rangle=p^{(0)}$, it follows that $\partial p^{(1)}/\partial\xi=0$. This means that $p^{(1)}$ does not depend on ξ. After integrating over the structure period the equations containing the value of zero order of ε, we obtain the averaged system

$$\frac{\partial(r^{\nu})^{(0)}}{\partial s}=\langle V^{(0)}\rangle,\qquad u^{(0)}=\frac{\partial r^{(0)}}{\partial t},$$

$$\frac{\partial u^{(0)}}{\partial t}+\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(0)}}{\partial s}=0,$$

$$\frac{\partial\langle V^{(0)}\rangle}{\partial t}-\nu\frac{\partial(r^{\nu-1})^{(0)}u^{(0)}}{\partial s}=0, \tag{6}$$

$$d\langle V^{(0)}\rangle=-\left\langle\frac{(V^{(0)})^2}{c_f^2}\right\rangle\mathrm{d}p-\left\langle\frac{V^{(0)}}{\tau V_e(p^{(0)})}(V^{(0)}-V_e(p^{(0)}))\right\rangle\mathrm{d}t. \tag{7}$$

Unlike the values $u^{(0)}$, $p^{(0)}$, $p^{(1)}$ and $r^{(0)}$, the specific volume $V^{(0)}$ is a function of ξ. Hereafter, we will consider only the zero approximation of the equations and, therefore, the upper index 0 is omitted.

The averaged system of Eqs. (6) and (7) is an integro-differential one and, in the general case, is not reduced to the averaged variables p, u and $\langle V\rangle$. The derivation of Eqs. (6) and (7) relates to a rigorous periodic medium. However, it may be shown that Eqs. (6) and (7) are also relevant for a media with a quasi-periodic structure.

Indeed, the pressure, p, and the mass velocity, u, are independent of the fast variable ξ. Hence on a microscale, ξ, the action is statically uniform (waveless) over the whole period of the medium structure, while on the slow scale, s, the action of perturbation is manifested by the wave motion of the medium. On a microlevel the medium's behavior adheres only to the thermodynamic laws. There is a mechanical equilibrium. On a macrolevel, the medium's motion is described by the wave dynamics laws for averaged variables. Mathematically, in the zero-order case of ε, the size of the period is infinitesimal ($\varepsilon\to 0$). This signifies that the location of particular components in the period is irrelevant. Eqs. (6) and (7) do not change their form if the components are broken and/or change their location in an elementary cell. This means that Eqs. (6) and (7) describe the motion of any quasi-periodic (statistical heterogeneous) medium which has a constant mass content of components on the microlevel, and the location of these components within the cell is not important.

In the case of non-linear wave propagation, the individual components suffer different compressions. The medium's structure is changed, with the result that the averaged specific volume $\langle V \rangle$ is changed. This change differs from the change of the specific volume for homogeneous medium under the same loading. Thus, the medium's structure is manifested in the wave motion, despite the fact that the equations of motion (6) are written down for the averaged values u, p, $\langle V \rangle$ only.

3. System of equations in Eulerian coordinates

In certain cases of theoretical analysis it is more convenient to use the Eulerian coordinate system. The immediate employment of the averaging asymptotic method in Eulerian variables is impossible because of the variability of the microstructure sizes. However, from the zero approximation in the equations of motion (3), which are presented by the averaged values p, u, $\langle V \rangle$, the equations can be rewritten in the Eulerian system of coordinates (r, t_E) by means of a transformation from the Lagrangian system (s, t).

There is an important presumption that the velocity of the particle in the zero approximation is constant over a period of the structure and, consequently, we can describe an averaged trajectory for the particle. From the physical point of view, it is clear that the position of the particle is unambiguously defined by its coordinates and time. From the mathematical point of view this means that in the transformation

$$\mathrm{d}r^{\nu} = A\,\mathrm{d}s + \nu r^{\nu-1} u\,\mathrm{d}t, \qquad t_E = t$$

the value $\mathrm{d}r^{\nu}$ is a total differential. Therefore, we must have

$$\frac{\partial A}{\partial t} = \frac{\partial \nu r^{\nu-1} u}{\partial s}.$$

This condition is satisfied if $A = \langle V \rangle$, because the equation converts to the continuity equation (6). We obtain the following transformation between Lagrangian and Eulerian systems of coordinates:

$$\mathrm{d}r^{\nu} = \langle V \rangle \mathrm{d}s + \nu r^{\nu-1} u\,\mathrm{d}t, \qquad t_E = t.$$

It is reasonable to define the slow Lagrangian coordinate the (non-mass one) as $R^{\nu} = s\langle V_0 \rangle$. Eq. (6) in the Eulerian system of coordinates then takes the form

$$\frac{\partial \langle V \rangle^{-1}}{\partial t_{\mathrm{E}}} + \frac{\partial u r^{\nu-1} \langle V \rangle^{-1}}{\partial r} = 0,$$

$$\frac{\partial u}{\partial t_{\mathrm{E}}} + u\frac{\partial u}{\partial r} + \langle V \rangle \frac{\partial p}{\partial r} = 0. \qquad (8)$$

It should be noted that the average density of a medium in Eulerian coordinates is $\tilde{\rho} = \langle V \rangle^{-1}$ [20] and the equations of motion (8) can be written in the usual form of the averaged density, $\tilde{\rho}$. The value $\tilde{\rho}$ is real density. The value $\langle V \rangle$ is the specific volume averaged in units of mass over the period. $\langle V \rangle$ is expressed as the ratio of the volume to the mass inside this volume. This value can be determined experimentally.

The method of the computer solution for the system of equations was described in [20, 21], where the integration step is restricted by the perturbation wavelength and not by the period of the structure. The main computational problem associated with the smallness of the integration step can be avoided, and the equations of motion can be solved large distance of wave propagation within a reasonable time. We have developed a software package for a computerized solution of the system of equations.

4. Acoustic waves

Let us consider an acoustic wave ($p' = p - p_0$, $p' \ll p_0$). The small pressure perturbations propagate similarly in periodic and homogeneous media. We shall prove that the propagation of the acoustic waves in a periodic medium with a calculable number of relaxation components is similar to that in a homogeneous medium with the same number of independent relaxation processes.

Now we shall show this for a two-layer periodic medium with one process of relaxation in each structure element. The averaged equation of state (7) for small perturbations in such a medium can be represented as

$$-\langle V'\rangle = \langle V^2/c_f^2\rangle p' + \kappa \frac{V_1^2(c_{1e}^{-2} - c_{1f}^{-2})}{1+\tau_{1\,per}\,d/dt} p' + (1-\kappa)\frac{V_2^2(c_{2e}^{-2} - c_{2f}^{-2})}{1+\tau_{2\,per}\,d/dt} p',$$

$$\langle V^2/c_e^2\rangle = \kappa V_1^2/c_{1e}^2 + (1-\kappa) V_2^2/c_{2e}^2, \tag{9}$$

where index 1 relates to the first component, and index 2 to the second component. Here κ is a coordinate of the boundary between the components in the elementary cell, and the values κ and $1-\kappa$ are equal to the mass concentration of the first and the second component, respectively.

For comparison we take the homogeneous medium with two independent relaxation processes. The equations state of such a medium for small perturbations has a form [24]

$$-V' = V^2/c_f^2 p' + \frac{V^2(c_{e1}^{-2} - c_{f1}^{-2})}{1+\tau_{1\,hom}\,d/dt} p' + \frac{V^2(c_{e2}^{-2} - c_{f2}^{-2})}{1+\tau_{2\,hom}\,d/dt} p', \quad c_f^{-2} = \sum_i c_{fi}^{-2}. \tag{10}$$

It should be noted that the alphanumeric indices for the homogeneous medium and for the periodic one have reverse succession. Here, index 1 relates to the first relaxation process, and index 2 to the second process.

Now we can write six relationships

$$\kappa_i V_i^2 (c_{ie}^{-2} - c_{if}^{-2})_{per} = V^2 (c_{ei}^{-2} - c_{fi}^{-2})_{hom},$$

$$\langle V^2/c_e^2\rangle = V^2 \sum_i c_{ei}^{-2}, \quad \langle V^2/c_f^2\rangle = (V^2/c_f^2)_{hom},$$

$$\tau_{i\,per} = \tau_{i\,hom}, \quad \kappa_1 = \kappa, \quad \kappa_2 = 1-\kappa, \quad i = 1, 2. \tag{11}$$

These equations show that for any two-component medium with both the relaxation components ($\tau_{i\,per}$, c_{ie}, c_{if}) [see Eq. (9)] we can pick up the homogeneous medium with two relaxation processes ($\tau_{i\,hom}, c_{ei}, c_{fi}$) [see Eq. (10)]. In such media perturbations $\langle V\rangle, p, u$ move in a similar way. Regarding the density $\langle\rho\rangle$ this statement is incorrect. The result can be, easily expanded on the media with a calculable number of the relaxation components. This result proves the statement that in the studies of acoustic wave propagation in a periodic medium with N relaxation components this medium can be substituted by a homogeneous one in which there are N independent relaxation processes.

The similarity of the propagation of a small perturbation in periodic and homogeneous media was verified numerically. As it was expected, we obtained the traditional result. An inner structure of the medium manifests itself only by means of the dispersive dissipative properties. For the acoustic level the long wave dynamic behavior of the medium with a microstructure can be modeled within the framework of a homogeneous relaxing medium. In the past such a statement was accepted *a priori*. In our case we have obtained a rigorous mathematical proof of this statement on the basis of a detailed accounting of the medium's structure.

5. Non-linear waves

We have analyzed the propagation of non-linear waves in a structured medium. To make the results more clear, we will restrict our consideration to a non-relaxing media ($c = c_f = c_e$). The averaged state equation in this case is simplified to the form

$$d\langle V\rangle = -\langle V^2/c^2\rangle\,dp, \tag{12}$$

and we can introduce an effective sound velocity by the formula

$$c_{eff} = \sqrt{\frac{\langle V\rangle^2}{\langle V^2/c^2\rangle}}. \tag{13}$$

We obtain a traditional representation of a system of Eqs. (6), (8) and (12).

Normalization on the averaged specific volume $\langle V\rangle$ and the initial sound velocity c_{eff} allows us to compare the results for various media. For convenience we have chosen that the acoustic waves in such a media propagate in a similar way [see Eq. (11)].

It should be noted that c_{eff} is not an averaged value, i.e. $c_{\mathrm{eff}}^2 \neq \langle c^2 \rangle$. Evidently, the structure of the medium introduces a certain contribution to the non-linearity. In fact, even if $c \neq f(p)$, then in the general case the value of c_{eff} is a function of pressure.

The system of Eq. (6) is hyperbolic and this specifies its breaking solutions, which are shock waves. For the analysis of such solutions, it is necessary to present Eq. (6) in the form of integral conservation laws,

$$\oint \langle V \rangle \, \mathrm{d}s + u \, \mathrm{d}t = 0, \quad \oint u \, \mathrm{d}s - p \, \mathrm{d}t = 0.$$

Now we can easily formulate the conditions on the shock front, when there is conservation of the fluxes of mass and of impulse through the frontier of the break

$$(\langle V_1 \rangle - \langle V_0 \rangle) D + u_1 - u_0 = 0,$$

$$(u_1 - u_0) D - p_1 + p_0 = 0,$$

where indexes 0 and 1 relate to the parameters of the flow before and after the front, respectively. Hence, the formula for the averaged velocity of the shock front in terms of the Lagrangian variable D (dimension $[D]$, kg/c) and the mass velocity u follow from the following relations:

$$D = \sqrt{(p_1 - p_0)/(\langle V_0 \rangle - \langle V_1 \rangle)},$$

$$u_1 - u_0 = \sqrt{(p_1 - p_0)(\langle V_0 \rangle - \langle V_1 \rangle)}.$$

Now we shall prove the statement that the medium's structure always increases the non-linear effects on the propagation of long waves. At first, let us consider the sound velocity in homogeneous c_{hom} and heterogeneous c_{eff} media. It has been found in the general case that with increasing pressure, the velocity of the sound is greater in a structured medium than in a homogeneous one. For the sake of clarity, we consider a medium in which the sound velocities of the separate components are independent of the pressure $c \neq f(p)$

$$c_{\mathrm{eff}} \geqslant c_{\mathrm{hom}}. \tag{14}$$

The equality sign is fulfilled (a) for an initial pressure, by virtue of the normalization, and also (b) for a special structured medium in which the relation $V(\xi)/c^2(\xi)$ is not a function on the fast variable ξ. We must prove which case results in an equality and which gives the inequality.

The derivative $\mathrm{d}c_{\mathrm{hom}}/\mathrm{d}p = 0$, and

$$\frac{\mathrm{d}c_{\mathrm{eff}}}{\mathrm{d}p} = \frac{2\langle V \rangle}{\langle V^2/c^2 \rangle} \left(\langle V \rangle \left\langle \frac{V^3}{c^4} \right\rangle - \left\langle \frac{V^2}{c^2} \right\rangle^2 \right) \geqslant 0.$$

This last inequality follows from the well-known Cauchy–Schwarz inequality (see, for instance, [29]). Therefore, with the increase of pressure, the sound velocity c_{eff} increases. Consequently, we have the inequality (14) at $p \geqslant p_0$.

Moreover, at $p > p_0$ the shock adiabatic curve for the medium with a structure always lies above that for the homogeneous one (they touch only at the initial point $p = p_0$)

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\langle V \rangle^2} \geqslant \left(\frac{\mathrm{d}^2 p}{\mathrm{d}V^2} \right)_{\mathrm{hom}}. \tag{15}$$

Indeed, a ratio of these derivatives is equal to

$$\frac{\mathrm{d}^2 p/\mathrm{d}\langle V \rangle^2}{(\mathrm{d}^2 p/\mathrm{d}V^2)_{\mathrm{hom}}} = \frac{\langle V^3/c^4 \rangle \langle V^2/c^2 \rangle^{-3}}{c_{\mathrm{hom}}^2 \langle V \rangle^3}$$

$$= \frac{\langle V^3/c^4 \rangle \langle V \rangle c_{\mathrm{eff}}^2}{c_{\mathrm{hom}}^2 \langle V^2/c^2 \rangle^2} \geqslant \frac{\langle V^3/c^4 \rangle \langle V \rangle}{\langle V^2/c^2 \rangle^2} \geqslant 1.$$

Hence, a long wave with a finite amplitude responds to the structure of the medium, and the non-linear effects increase as compared to the homogeneous medium. The non-linearity takes place, even if separate components are described by the linear law.

An exception, as it was noted already, is for a medium with the properties of structure so that $V(\xi)/c^2(\xi) \neq f(\xi)$. For this medium only the equality sign is correct in inequalities (14) and (15). Particular elements of the structure respond to the pressure variations, and this means that the relative structure does not change, i.e. the ratio $V(\xi, p)/V(\xi, p_0)$ does not depend on ξ. In this case, the value of $c_{\mathrm{eff}} = \sqrt{\langle c^2 \rangle}$ is an averaged characteristic [see Eq. (13)]. Therefore, the system of equations may be presented using the averaged variables p, u, $\langle V \rangle$, $c_{\mathrm{eff}} = \sqrt{\langle c^2 \rangle}$. Heterogeneity does not introduce an additional non-linearity for this medium, and the structure of medium does not affect the wave motion.

6. Lyakhov's model as special case of an asymptotic averaged model

The derivation of the averaged equations of motion (6) and (8) gives the rigorous mathematical foundation for the use of one-velocity continuous models for a heterogeneous medium, and these models are asymptotic in nature.

Over several years this last model has been successfully applied to describe explosions and non-linear waves in soil [16, 22].

Experiments which have been conducted over a number of years on the action of explosive and shock loading on the soft soil (see Ref. [30]) show an effect of time lag of the strain with respect to the stress (relaxation effect) and large deformations of the soil. The Lyakhov's model enables us to take into account all these effects and describe the propagation of non-linear waves (explosion, shock waves) in soil and also in a bubble media [16]. In this approach the multicomponent medium is considered as a homogeneous continuum with a special equation of state. The macrovolume includes all the components that compose the medium. The average values – pressure p_L, specific volume V_L, mass velocity u_L are defined in the normal way. The equations of motion have the following form in the Lagrangian coordinate system [16]:

$$\frac{\partial V_L}{\partial t} - \nu V_{L0}\frac{\partial r_L^{\nu-1}u_L}{\partial R^\nu} = 0, \quad u_L = \frac{\partial r_L}{\partial t},$$

$$\frac{\partial u_L}{\partial t} + V_{L0}\left(\frac{r}{R}\right)^{\nu-1}\frac{\partial p_L}{\partial R} = 0. \tag{16}$$

There is a direct connection between Lyakhov's model and the averaged Eqs. (6) and (7). First we will show that the variable V_L introduced by Lyakhov is nothing more than the averaged value of the specific volume in the mass Lagrangian coordinate $\langle V\rangle$. In order to do this we will start by making the following transformation. The value of a specific volume V_L is expressed through the initial volume content of the ith component α_i by the formula (in agreement with p. 56 of Ref. [16] $V_L = V_{L0}\sum_{i=1}^{3}\alpha_i V_i/V_{i0}$. It is easily seen that

$$V_L = V_{L0}\sum_{i=1}^{3}\beta_i\frac{V_{i0}}{V_{L0}}\frac{V_i}{V_{i0}} = \sum_{i=1}^{3}\beta_i V_i = \langle V\rangle,$$

where we used the connection between the parameter α_i and the size of ith component β_i (in mass Lagrangian coordinates)

$$\alpha_i = \beta_i\frac{V_{i0}}{V_{L0}}. \tag{17}$$

The value β_i is the mass content of the ith component. Obviously, β_i is constant and independent of the wave motion.

As shown earlier in the asymptotic averaging method for zero-order, the pressure and mass velocity are independent of the fast coordinate ξ. At the same time in Lyakhov's model it is suggested, *a priori*, that the pressure in all components is equal, and that they move with equal velocity. Comparing the equations of motion (6) and (16), we can see that there are connections between the values of pressure $p_L = p^{(0)}$, the values of mass velocities $u_L = u^{(0)}$ and the values of Eulerian space coordinates $r_L = r^{(0)}$. Hereafter, the index L in p, u and r is omitted.

The microstructure is taken into account by means of the dynamic equation of state in Lyakhov's model [16] by

$$\frac{\dot V_L}{V_{L0}} = \varphi(p)\dot p - \frac{\alpha_1}{\eta}\psi(p, V_L), \tag{18}$$

$$\varphi(p) = -\sum_{i=2}^{3}\frac{\alpha_i}{\rho_{i0}c_{i0}^2}\left[\frac{\gamma_i(p-p_0)}{\rho_{i0}c_{i0}^2}+1\right]^{-(1+\gamma_i)/\gamma_i}, \tag{19}$$

$$\psi(p, V_L) = p - p_0 - \frac{\rho_{10}c_{10}^2}{\gamma_1} \times\left[\left\{\frac{V_L}{V_{L0}} - \sum_{i=2}^{3}\alpha_i\left[\frac{\gamma_i(p-p_0)}{\rho_{i0}c_{i0}^2}+1\right]^{-1/\gamma_i}\right\}^{-\gamma_1} \times\alpha_1^{\gamma_1} - 1\right].$$

Here the index number 0 relates to the initial non-perturbed state; 1 corresponds to the parameters of air; 2 to the parameters water; 3 to the solid substance; α_i is the initial volume content of the ith phase in soil; $\sum_{i=1}^{3} \alpha_i = 1$; and γ_i is the exponent in the Tait's equations of state. In Lyakhov's equations of state, an important role is played by the coefficient of the medium's volume viscosity η, which in the general case depends on the pressure and the specific volume $\eta = \eta(p, V_L)$.

At $\dot{p} = 0$ and $\dot{V}_L = 0$ we obtain the following equation of the equilibrium compressibility medium:

$$\frac{(V_L)_e}{V_{L0}} = \sum_{i=1}^{3} \alpha_i \left[\frac{\gamma_i (p - p_0)}{\rho_{i0} c_{i0}^2} + 1 \right]^{-1/\gamma_i}. \tag{20}$$

The sound velocity in such a process can be found by the formula [16]

$$c_L = \frac{\sum_{i=1}^{3} \alpha_i \left[\frac{\gamma_i (p - p_0)}{\rho_{i0} c_{i0}^2} + 1 \right]^{-1/\gamma_i}}{\left\{ \rho_0 \sum_{i=1}^{3} \frac{\alpha_i}{\rho_{i0} c_{i0}^2} \left[\frac{\gamma_i (p - p_0)}{\rho_{i0} c_{i0}^2} + 1 \right]^{-(1+\gamma_i)/\gamma_i} \right\}^{1/2}}. \tag{21}$$

Let us now compare the equations of state (7) and (18). We shall prove that the equation of state (18) in Lyakhov's model implies that the medium cannot be considered as homogeneous, and that this equation of state is an averaged equation.

At first we shall consider the non-relaxing medium. In this case the equation of state (18) has the form of Eq. (20) and it can be shown that expressions (20) and (12) coincide if in Eq. (7) the dependence of the sound velocity on the pressure is concretely defined by utilizing Tait's relationship as in Eq. (20). Therefore, we must check the case $p \to p_0$. The substitution of expression (17) in Eq. (21) gives

$$c_{L0} = \left[\rho_0 \sum_{i=1}^{3} \frac{\alpha_i}{\rho_{i0} c_{i0}^2} \right]^{-1/2} = \rho_0^{-1} \left[\sum_{i=1}^{3} \beta_i \frac{V_{i0}^2}{c_{i0}^2} \right]^{-1/2} = \langle V_0 \rangle \left\langle \frac{V_0^2}{c_0^2} \right\rangle^{-1/2} = (c_{\text{eff}})_0.$$

This value consists of an effective averaged sound velocity for the periodic medium (13) and thus expression (21) changes to Eq. (12).

Let us take into account the processes of relaxation and consider the dynamic equation of state in Lyakhov's model (18), (19). For weak perturbations we have

$$\psi(p, V_L) = \Delta p - \frac{\rho_{10} c_{10}^2}{\gamma_1} \times \left[\left\{ \frac{V_L}{V_{L0}} - \sum_{i=2}^{3} \alpha_i \left(\frac{\gamma_i \Delta p}{\rho_{i0} c_{i0}^2} + 1 \right)^{-1/\gamma_i} \right\}^{-\gamma_1} \alpha_1^{\gamma_1} - 1 \right]$$
$$= \Delta p - \frac{\rho_{10} c_{10}^2}{\gamma_1} \left[\left\{ \frac{V_L - (V_L)_e}{V_{L0}} + \alpha_1 \left(\frac{\gamma_1 \Delta p}{\rho_{10} c_{10}^2} + 1 \right)^{-1/\gamma_1} \right\}^{-\gamma_1} \alpha_1^{\gamma_1} - 1 \right]$$
$$= \Delta p - \frac{\rho_{10} c_{10}^2}{\gamma_1} \left(-\gamma_1 \frac{V_L - (V_L)_e}{\alpha_1 V_{L0}} + \frac{\gamma_1 \Delta p}{\rho_{10} c_{10}^2} \right) = \rho_{10} c_{10}^2 \frac{V_L - (V_L)_e}{\alpha_1 V_{L0}}.$$

where $(V_L)_e$ is defined by Eq. (20). The equation of state takes the form

$$\frac{\dot{V}_L}{V_{L0}} = \varphi(p) \dot{p} - \frac{\rho_{i0} c_{i0}^2}{\eta} \frac{V_L - (V_L)_e}{V_{L0}}. \tag{22}$$

We will now consider the value of φ in Eq. (19). The sum is calculated for the solid components ($i = 2, 3$) and this signifies that high-frequency sound velocity of a multicomponent medium is determined by the sound velocity in solid components. The gas phase is considered as incompressible for these perturbations [16] ($c_{1f} \to \infty$). We will now restore this term, and take the high-frequency sound velocity c_{1f} equal to the velocity of the solid phase component and under the condition $p \to p_0$ we can obtain

$$\varphi(p) = -\sum_{i=2}^{3} \frac{\alpha_i}{\rho_{i0} c_{i0}^2} = -\sum_{i=1}^{3} \frac{\alpha_i}{\rho_{i0} c_{if}^2} = -\langle V_0 \rangle^{-1} \langle V^2 / c_f^2 \rangle_0. \tag{23}$$

Finally, when we use the notation of the model of asymptotic averaging $V_L = \langle V \rangle$ together with Eq.

(23) Lyakhov's equation of state (22) has the following form:

$$\langle \dot{V} \rangle = -\langle V^2/c_f^2 \rangle \dot{p} - \tau^{-1}(\langle V \rangle - \langle V_e \rangle), \tag{24}$$

where $\tau = \eta^{-1}\rho_{10}c_{1e}^2$.

Thus, we see that expression (24) is the averaged equation of state. It does not reduced to the averaged variables $p, u, \langle V \rangle, \tau$, because of the term $\langle V^2/c_f^2 \rangle$.

Let us compare expression (24) with the dynamic equation of state for a periodic medium (7). In a periodic medium the relaxation processes are considered to occur in each component. Eq. (7) becomes expression (24) only when one component is relaxed, whereas in Lyakhov's model, the dependence of sound velocity on pressure is concretely defined.

Thus, a rigorous mathematical analysis has shown that for both the models the equations of motion are written in terms of averaged values $p, u, \langle V \rangle$, while the properties of separate components are contained in the equations of state. It is shown that the equations of motion coincide completely. The dynamic equations of state of the averaged description for these models also coincide in the sense that only one relaxation process is considered in the Lyakhov's model. However, in Lyakhov's model the dependence of sound velocity on the pressure is defined concretely, and the gas phase is considered incompressible for high-frequency perturbations. Our approach explains these details in a rigorous manner. Therefore, using an asymptotic method Lyakhov's model has been directly verified. Thus, it is proved that Lyakhov's model is asymptotic in nature.

7. Conclusions

Thus, in this work we present an asymptotic averaged model to explain the propagation of long non-linear waves in a non-equilibrium medium with a regular structure. In the general case, the averaged system of equations is integrodifferential and does not reduce to the averaged variables p, u and $\langle V \rangle$. On a microstructure level of the medium the dynamic behavior adheres only to the thermodynamic laws. On a macro level the motion of the medium can be described by wave dynamic laws for averaged variables with an integrodifferential equation of state containing the characteristics of a medium's microstructure. The suggested model justifies the one-velocity continuous model. A comparison of this model was carried out with Lyakhov's model for natural multicomponent media. We have shown that Lyakhov's model is asymptotic in nature. A rigorous mathematical proof is provided for the statement that on an acoustic level for long waves the inner structure of the medium manifests itself only by means of the dispersive dissipative properties, and the dynamic behavior of the medium can be modelled in the framework of a homogeneous relaxing medium. However, the long wave with a finite amplitude responds to the structure of the medium so that the behavior of the structured medium cannot be modeled by a homogeneous medium. An important result foretold by this model is that the medium structure always increases the non-linear effects on the long waves, and that non-linearity takes place even if separate components are described by a linear law.

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