LONG-WAVE PROCESSES IN A PERIODIC MEDIUM

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The development of experimental techniques has shown that the internal structure of a medium affects wave motion and transport processes. Nonlinear effects start playing an increasing role with increasing wave amplitude. To the same class belong high-gradient, fast-flowing nonequilibrium processes. All these features must be taken into account in the mathematical simulation of waves in real media.

There exists a class of effects, in which the characteristic size considered L is much larger than the size of medium inhomogeneity ε ($L \gg \varepsilon$). The drawback of assigning microproperties of an inhomogeneous medium creates major difficulties for direct solution of these problems. Numerical solutions are difficult due to high cost of computer time. One way of investigating an inhomogeneous medium is the spatial averaging method.

The averaged description has an asymptotic nature. There exist different mathematical methods of asymptotic averaging of long-wave processes with detailed account of the structure. In the present study, for wave modeling in a barotropic periodic medium we use the asymptotic averaging method developed for a composite regular structure [1, 2]. Processes in an inhomogeneous medium can be described by equations with fast-oscillating coefficients. For media of regular structure the fast-oscillating coefficients are periodic functions. The essence of asymptotic averaging consists of applying the multiple scale method [3] in conjuction with spatial averaging [4]. The method provides an asymptotically correct approximation to the solution. In the general case one obtains a system of integrodifferential equations. Sometimes the problem can be reduced to the average characteristics of wave fields. At the same time, in solving the integrodifferential system one can find a numerical method in which one succeeds in selecting a step in the spatial coordinate substantially exceeding the period of the structure.

1. System of Averaged Equations. The asymptotic averaging method can be applied to describe processes in a compressible inhomogeneous medium [5]. The equations of motion are conveniently written in Lagrangian coordinates. In these variables the structure of the periodic compressible medium is nonvarying, making it possible to use the averaging procedure.

In the general case the coefficients and solutions of the equations, describing processes in inhomogeneous media, have discontinuous values. The equations must then be represented in integral form. Under the assumption of smooth coefficients and solutions they are equivalent to the differential equation of motion. It was shown in [1, 6] that for insignificant differences in physical properties of medium inhomogeneity the approximate continuous solutions of the system of differential equations formally constructed by means of the asymptotic method satisfy the integral conservation laws with a certain accuracy. This fact indicates the correctness of using a system of differential equations of motion in an asymptotic averaging method.

The original equations of motion are the equations of a barotropic periodic medium, written down for each structural element [7]:

$$\frac{\partial V}{\partial t}-\frac{\partial u}{\partial m}=0,\ \frac{\partial u}{\partial t}+\frac{\partial p}{\partial m}=0. \tag{1.1}$$

Here $V \equiv \rho^{-1}$ is the specific volume, u is the mass velocity, ρ is the density, p is the pressure, m is the mass variable, and t is time.

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With the purpose of taking into account internal processes in the inhomogeneous medium it is assumed that each component manifests relaxation effects, including the first-order dynamic equation of state [8]

$$d\rho = c_f^{-2}\,dp - (\rho - \rho_e)\,\tau^{-1}\,dt, \tag{1.2}$$

where τ is the relaxation time, c_e and c_f are the equilibrium and frozen sound velocities, $\rho_e = \rho(p)$ is the equilibrium equation of state, while, more precisely, $\rho_e = \rho_0 + \int\limits_{p_0}^{p} c_e^{-2}\,dp$; and p_0, ρ_0 are the pressure and density in an arbitrary equilibrium state. Equation (1.2) covers the limiting cases of both the totally frozen and the equilibrium process. It includes the description of processes without relaxation with formal replacement of the relaxation time $\tau = 0$ by some nonvanishing quantity and under the condition $c_e = c_f$. In the following it is assumed that the sound velocity and the relaxation time do not depend on time explicitly, but only on pressure and on the individual properties of the medium components. This implies that the relaxation process inside the structural elements of the periodic medium leads only to momentum exchange without mass exchange.

We apply the asymptotic averaging method [1] to Eqs. (1.1), (1.2). According to the multiple scale method, the spatial variable m is decomposed into slow s and fast ξ independent variables:

$$m = s + \varepsilon\xi, \quad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{1.3a}$$

The slow variable corresponds to the global field structure, and the fast – to the local structure. The solution p, u, V is sought in the form of power series in the structural period ε:

$$p(m, t) = p(s, \xi, t) = p^{(0)}(s, \xi, t) + \varepsilon p^{(1)}(s, \xi, t) + \varepsilon^2 p^{(2)}(s, \xi, t) + \ldots \tag{1.3b}$$

A feature of the problem consists of the fact that due to the constant structure of the medium in Lagrangian coordinates the functions in the right hand side of series (1.3b) are assumed periodic in ξ. This leads to simplified equations by means of using the averaging procedure over the structural period.

Following substitution of (1.3) into (1.1) we obtain

$$-\varepsilon^{-1}\frac{\partial u^{(0)}}{\partial \xi} + \varepsilon^{0}\left(\frac{\partial V^{(0)}}{\partial t} - \frac{\partial u^{(0)}}{\partial s} - \frac{\partial u^{(1)}}{\partial \xi}\right) + \varepsilon^{1}\left(\frac{\partial V^{(1)}}{\partial t} - \frac{\partial u^{(1)}}{\partial s} - \frac{\partial u^{(2)}}{\partial \xi}\right) + \ldots = 0,$$

$$\varepsilon^{-1}\frac{\partial p^{(0)}}{\partial \xi} + \varepsilon^{0}\left(\frac{\partial u^{(0)}}{\partial t} + \frac{\partial p^{(0)}}{\partial s} + \frac{\partial p^{(1)}}{\partial \xi}\right) + \varepsilon^{1}\left(\frac{\partial u^{(1)}}{\partial t} + \frac{\partial p^{(1)}}{\partial s} + \frac{\partial p^{(2)}}{\partial \xi}\right) + \ldots = 0.$$

According to the general theory of the asymptotic method, terms with equal powers of ε must vanish independently of each other. Therefore, the equation with ε^{-1} are of the form $\partial p^{(0)}/\partial\xi = 0$, $\partial u^{(0)}/\partial\xi = 0$, providing independence of $p^{(0)} = p^{(0)}(s, t)$ and $u^{(0)} = u^{(0)}(s, t)$ on ξ. Besides, for terms with ε^0 one must have

$$\frac{\partial V^{(0)}}{\partial t} - \frac{\partial u^{(0)}}{\partial s} - \frac{\partial u^{(1)}}{\partial \xi} = 0, \quad \frac{\partial u^{(0)}}{\partial t} + \frac{\partial p^{(0)}}{\partial s} + \frac{\partial p^{(1)}}{\partial \xi} = 0.$$

We average over the period ε. By definition $\langle\cdot\rangle = \int\limits_0^1 (\cdot)\,d\xi$. Due to the periodicity of $p^{(1)}$ and $u^{(1)}$ the integrals satisfy $\langle \partial u^{(1)}/\partial\xi\rangle = 0$, $\langle \partial p^{(1)}/\partial\xi\rangle = 0$. And since $\langle u^{(0)}\rangle = u^{(0)}$, $\langle p^{(0)}\rangle = p^{(0)}$, then $\partial p^{(1)}/\partial\xi = 0$ while due to the periodicity of $p^{(1)}(\xi)$ the function $p^{(1)}$ is independent of ξ, Consequently, the averaged equations of motion (1.1) are, within the zeroth order in ε, [9]

$$\frac{\partial \langle V^{(0)}\rangle}{\partial t} - \frac{\partial u^{(0)}}{\partial s} = 0, \quad \frac{\partial u^{(0)}}{\partial t} + \frac{\partial p^{(0)}}{\partial s} = 0. \tag{1.4}$$

The quantities V, ρ, τ, c_e, c_f depend in the general case on the fast variable ξ, while the last three functional dependences are assumed to be given ahead of time. Following averaging, the equation of state (1.2) for the vanishing approximation in ε can be written formally as

$$\langle V_0^{(0)}\rangle - \langle V^{(0)}\rangle = \left\langle \frac{\tau c_f^{-2}\frac{dp^{(0)}}{dt} + \int\limits_{p_0}^{p^{(0)}} c_e^{-2}\,dp^{(0)}}{\left(1+\tau\frac{d}{dt}\right)\frac{1}{V^{(0)}V_0^{(0)}}} \right\rangle. \quad (1.5)$$

The average system (1.4), (1.5) is obtained, describing within the vanishing approximation in ε processes in a barotropic, periodic, relaxing medium. The structural characteristics appear only in Eq. (1.5).

We note that the asymptotic method was applied earlier for the averaged system of equations, describing compressible media with viscosity [5]. In this case the introduction of viscosity makes it possible, from a physical point of view, to describe only low-frequency perturbations. Besides, the system of averaged equations does not provide the possibility of investigating wave motions if none of the components of the periodic medium possesses a viscosity. A more complicated method of constructing averaged equations is suggested in [10], where an integrodifferential system with a lag is obtained, not containing a fast variable. However, it is indicated there that there exist no reliable methods of solving it.

2. Analysis of Propagating Pressure Waves. We investigate several averaged properties of sound waves in a periodic relaxing medium. We represent the unknown variables in the form

$$p = p_0 + p_1, \quad \rho = \rho_0 + \rho_1, \quad V = V_0 + V_1,$$

where p_0, ρ_0, V_0 are the unperturbed equilibrium values of pressure, density, and specific volume, while p_1, ρ_1, V_1 are their increments in a sound field, with $p_1 \ll p_0$, $\rho_1 \ll \rho_0$, $V_1 \ll V_0$. In what follows we omit the subscript 0, denoting the first term of the asymptotic expansion (1.3b). In this case the system (1.4), (1.5) acquires the following form within the linear approximation

$$\frac{\partial\langle V_1\rangle}{\partial t} - \frac{\partial u}{\partial s} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial p_1}{\partial s} = 0; \quad (2.1)$$

$$\langle V_1\rangle = -\left\langle V_0^2 \frac{c_e^{-2} + \tau c_f^{-2}\frac{d}{dt}}{1+\tau\frac{d}{dt}} \right\rangle p_1. \quad (2.2)$$

In the two limiting cases of high-frequency ($\omega\tau \gg 1$) and low-frequency ($\omega\tau \ll 1$) wave perturbations the averaged equation of state (2.2) are rewritten, respectively, as

$$\langle \dot V_1\rangle = -\left\langle \frac{V_0^2}{c_f^2}\right\rangle \dot p_1 - \left\langle V_0^2 \frac{c_f^2 - c_e^2}{\tau c_e^2 c_f^2}\right\rangle p_1, \quad \langle V_1\rangle = -\left\langle \frac{V_0^2}{c_e^2}\right\rangle p_1 + \left\langle \tau V_0^2 \frac{c_f^2 - c_e^2}{c_e^2 c_f^2}\right\rangle \dot p_1.$$

In this case we used the estimate $\omega \sim \psi^{-1}d\psi/dt$, where $\psi = p$ or V. The last equation makes it possible to interpret the expression

$$\zeta = \left\langle \tau V_0^2 \frac{c_f^2 - c_e^2}{c_e^2 c_f^2}\right\rangle \left\langle \frac{V_0^2}{c_e^2}\right\rangle^{-2}$$

as the averaged bulk viscosity coefficient for low-frequency perturbations.

For a nonrelaxing periodic medium the equation of state is simplified:

$$d\langle V\rangle = -\langle V_0^2/c^2\rangle\, dp. \quad (2.3)$$

The system (2.1), (2.3) is expressed in terms of averaged characteristics. By comparing it with the original equations for a homogeneous medium it follows that the pressure field in a homogeneous nonrelaxing medium coincides with the average field in a periodic medium if $V_0^2/c^2 = \langle V_0^2/c^2 \rangle$. In the special case of a periodic medium with two components, one of which is relaxing, the average pressure perturbation propagates as in a homogeneous relaxing medium with the consistency conditions

$$V_0^2/c_e^2 = \langle V_0^2/c_e^2 \rangle,\ V_0^2/c_f^2 = \langle V_0^2/c_f^2 \rangle,\ \tau = \langle \tau \rangle. \tag{2.4}$$

In the limiting frozen and equilibrium processes the perturbations in a periodic relaxing medium behave as they do in some homogeneous nonrelaxing media.

It is possible to obtain a partial solution of the averaged integrodifferential equations without restrictions on the perturbation value for a medium with nonrelaxing components. The averaged system of equations for such a periodic medium (2.1), (2.3) is hyperbolic. The equations of characteristics in a Lagrangian coordinate system (the massive spatial coordinates) are

$$ds/dt = \pm \langle V^2/c^2 \rangle^{-1/2}, \tag{2.5}$$

on their valid Riemann invariants

$$J_\pm = u \pm \int \langle V^2/c^2 \rangle^{1/2}\, dp. \tag{2.6}$$

Expression (2.5) is the average perturbation propagation velocity in Lagrangian coordinates, depending on pressure and on the structure.

For a barotropic medium the pressure p and the velocity u in the wave can be expressed in the form of a function of each other. Equation (2.5) is then integrated

$$s = \pm t \langle V^2/c^2 \rangle^{-1/2} + f(u)$$

(f(u) is an arbitrary function of u). The relation given above, along with (2.6) describes the average behavior of a simple wave in a periodic barotropic medium. In the special case f(u) = 0 the solution is self-similar.

The structure of the medium directly affects the propagation of a simple wave of substantial amplitude even in the long-wave approximation. Equations (2.5), (2.6) contain the nonlinear terms $\langle V^2/c^2 \rangle$, depending on pressure and on the structure of the medium. Consequently, the average behavior of the pressure wave cannot now reduce to the behavior in a homogeneous medium.

As an example we show in Fig. 1 the pressure profile in a self-similar dilation wave, generated in a periodic medium while advancing a piston with constant velocity. The ratio of initial pressure in the medium p_{20} to the piston pressure p_{10} was taken to be $p_{20}/p_{10} = 10$. The elementary unit cell of the periodic medium was represented by two layers of identical size in Lagrangian coordinates with densities $\rho_{01} = 1.5\langle\rho\rangle$, $\rho_{20} = 0.5\langle\rho\rangle$ at pressure p_{10} ($\langle\rho\rangle$ is the mean density of the medium at this pressure). The sound velocity in both components was assumed identical and constant $(c = 1.5\sqrt{p_{10}/\langle\rho\rangle})$. The mass velocity was consistent with the pressure according to (2.6):

$$u = \int\limits_{p_{20}}^{p} \sqrt{\langle V^2 \rangle/c^2}\, dp.$$

The dimensionless dependences of the pressure p/p_{10} on the Lagrangian mass coordinate $\eta = \frac{s}{\tau_0}\sqrt{\langle\rho(p_{10})\rangle p_{10}}$ (τ_0 is the characteristic time) are shown for various moments of time in Fig. 1, where the solid lines are profiles calculated by the numerical-asymptotic method (discussed later) for a periodic medium, and being in satisfactory agreement with the dashed lines, constructed from the analytic solution for a periodic medium. With the purpose of comparison we show by dashed-dotted lines in Fig. 1 the profiles for a homogeneous medium, in which $\rho = \langle\rho\rangle$. The deviation of dilatation wave profiles in a periodic medium from a linear dependence is caused by its structure.

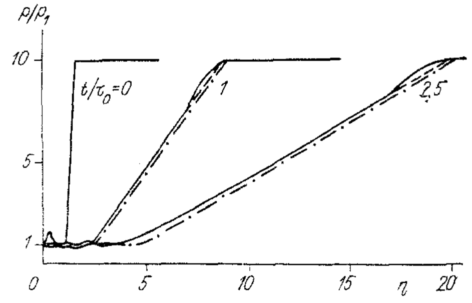

Fig. 1

It is interesting to note that the wave profile contains much information about the structure of the medium. In the special case in which the sound velocity in the medium is unique and constant, the structure of the medium can be established from the features of the propagating dilatation wave. In this case it must be understood that in the long-wave model the period is infinitely short, and, consequently, the location of structural elements in the period cannot be indicated accurately. We therefore select a medium in which the function $V = V(\xi)$ is a nonincreasing integrable one-to-one function. Multiplying both sides of the equation of state $dV = -V^2 c^{-2} dp$ by V^n (n is a natural number), following averaging we have $d\langle V^{n+1}\rangle/dp = -(n+1)c^{-2}\langle V^{n+2}\rangle$. The equation for the characteristics (2.5) makes it possible to find the function $\langle V^2\rangle(p)$. The equation derived provides the remaining $\langle V\rangle$, $\langle V^3\rangle$, $\langle V^4\rangle$, $\langle V^5\rangle$, ..., $\langle V^n\rangle$....

From probability theory [11] it is well known that the distribution f(x) (an arbitrary single-valued integrable positive function, defined over the whole x-axis) is related to its moments $\alpha_n = \int\limits_{-\infty}^{\infty} x^n f(x)\,dx$ by means of the characteristic function $\chi(q) = F[f(x)](q)$, where $F[\cdot]$ is the Fourier transform. Besides, the characteristic function is expressed in terms of the moments:

$$\chi = \sum_{n=0}^{\infty} \alpha_n i^n \frac{q^n}{n!}.$$

Taking into account the restrictions on the function $V(\xi)$, we write $\langle V^n\rangle = n\int\limits_0^{V(0)} \xi(V)\,dV$.

Assuming that the function $\xi(V)$ vanishes outside the segment $[0, V(0)]$, the last expression can be related to the central moment of the distribution function $f(x) \equiv \xi(V)$:

$$\langle V^n\rangle = n\int\limits_{-\infty}^{\infty} V^{n-1}\xi(V)\,dV = n\alpha_{n-1}.$$

Hence, by the inverse Fourier transform of the characteristic function we finally have

$$\xi(V) = F^{-1}\left[\sum_{n=0}^{\infty} \frac{\langle V^{n+1}\rangle}{(n+1)!} i^n q^n\right](V).$$

Thus is derived the inverse function of the required one, and the structure of the medium is found with the indicated accuracy.

3. The Numerical-Asymptotic Method. The equations of motion (1.4) have been written down in the averaged characteristics p, u, $\langle V\rangle$, depending on the slow variable s and on time t. The equation of state (1.5) is an integrodifferential equation with parameters depending on both slow and fast variables. The method of searching a solution of the system of equations (1.4), (1.5) is not obvious.

We describe a possible method of reducing the equations to a form in which the required functions depend only on the slow variable and on time. All functions depending on ξ are represented in the form of Fourier series on a segment corresponding to a period of the structure (for example, $\rho(\xi) = \sum_{k=-\infty}^{\infty} \rho_k \exp(2\pi ik\xi)$). The equations of motion remain unchanged. They are supplemented by relations for the series coefficients V_k and ρ_k, following from the equations of state and $\rho V = 1$. Both sides of these equations, expressed by Fourier series, are multiplied successively by $\exp(2\pi ik\xi)$ $(k = 0, \pm 1, \pm 2, \pm 3, \ldots)$ and are determined from the period of the structure. As a result of the two equations one forms an infinite chain. The same equations are represented by double trigonometric series, which can be represented schematically in the form

$$\begin{gathered}(c_f^{-2})_k\dot{p} - \dot{\rho}_k + \sum_{n=-\infty}^{\infty} (c_e^{-2})_n(\tau^{-1})_{k-n}(p - p_0) + (\tau^{-1})_{k-n}(\rho_{0n} - \rho_n) = 0,\\ \sum_{n=-\infty}^{\infty} (\rho_n V_{-n})_k = \delta_{0k},\ k = 0, \pm 1, \pm 2, \ldots,\end{gathered} \tag{3.1}$$

where the k-th terms are the Fourier series coefficients of the corresponding functions ρ, ρ_0, V, V_0, c_f^{-2}, c_e^{-2}, τ^{-1}.

An inifinite system of equations (1.4), (3.1) was obtained in p, u, ρ_k, V_k, which are functions of s and t. The density ρ and the specific volume V, depending on the fast variable ξ, are obtained following calculations of sums of Fourier series. In numerical calculations one can confine oneself to partial series sums, while the system of equations is closed. The process is described with the same accuracy with which the restricted Fourier series reproduces the structure of the medium. This makes it possible to separate the averaged problem in the slow variable, and in the computer solution select the step in the spatial coordinates by the perturbation wavelength and not by the period of the structure. The basic difficulty is thus overcome, and the wave propagation can be found at large distances.

In the numerical experiments the period of the structure did not impose any restriction on the spatial step. However, the use of partial Fourier series leads to a substantial increase in the number of equations (3.1) including nonlinear factors of the type $\rho_k V_n$.

The system (1.4), (3.1) was solved numerically by a finite-difference method in dimensionless quantities. From (1.4) one finds $\langle V \rangle$ and u by an explicit two-layered difference scheme. The pressure p and the Fourier coefficients ρ_k, V_k at the following temporal layer are determined by solving the 4m + 2 nonlinear equations (3.1) by the iterative Newton method (2m + 1 is the number of terms in the partial Fourier series sum). At each iteration the equations are transformed to linear, and the system obtained is solved by the Gauss method. The algorithm suggested was implemented in a Fortran program written for BÉSM-6 with invoking mathematical control by computer.

Besides calculating the motion of a self-similar dilatation wave in a periodic medium (Fig. 1), we calculated the propagation of acoustic perturbations in relaxing media. The structure of the periodic medium also consisted of two layers of equal extension with unperturbed medium parameters

$$\rho_0(\xi) = \begin{cases} 1{,}5\rho_0 \\ 0{,}5\rho_0 \end{cases},\ c_f = \begin{cases} 1{,}5\sqrt{p_0/\rho_0} \\ 1{,}5\sqrt{p_0/\rho_0} \end{cases},\ c_e = \begin{cases} c_f, & 0 < \xi \leqslant 0{,}5 \\ 0{,}1c_f, & 0{,}5 < \xi \leqslant 1{,}0. \end{cases}$$

The initial perturbation parameters (mass velocity, pressure, density) were mutually related in terms of the sound velocity c_f:

$$u = p_1 c_f^{-1}\sqrt{\langle V_0^2(\xi)\rangle},\quad \rho_1(\xi) = p_1/c_f^2.$$

Initial conditions were assigned so as to guarantee invariance in the pressure profile with time in media without relaxation. The sound velocity in a homogeneous medium was selected with condition (2.4) satisfied, i.e., $c_f = 1.006\sqrt{p_0/\rho_0}$, $c_e = 0.106\sqrt{p_0/\rho_0}$. The pressure sound wave must then operate identically in homogeneous and periodically relaxing media. The numerical calculations have verified these conditions and served as tests of the software developed. Estimating the characteristic perturbation frequency ω in terms of the halfwidth of the initial perturbation and of the frozen sound velocity c_f, $\omega\tau$ can be determined. The $\omega\tau$ value was varied for different relaxation times, making it possible to investigate the wave evolution as a function of the relations between the dynamic and relaxation characteristics.

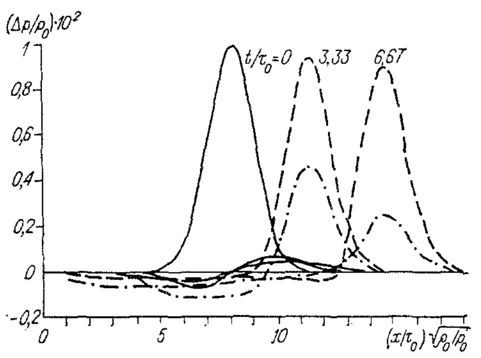

Fig. 2

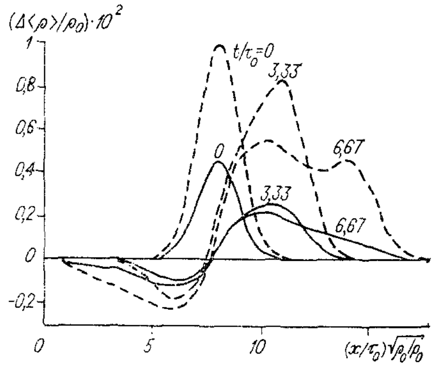

Fig. 3

The pressure dependence on the dimensionless Euler coordinate $\varkappa = x\tau_0^{-1}\sqrt{\rho_0 p_0^{-1}}$ is shown in Fig. 2 for relaxing media. The initial pressure perturbation in the form of a Gaussian distribution, whose center is at $\varkappa = 8$, are identical. The dashed curves are constructed for the dimensionless relaxation time $\tau/\tau_0 = 2000$ ($\omega\tau \simeq 30$), the dashed-dotted – for $\tau/\tau_0 = 200$ ($\omega\tau \simeq 3$), and the solid ones – for $\tau/\tau_0 = 20$ ($\omega\tau \simeq 0.3$). The pressure perturbation profiles are shown for the initial moment of time, as well as for $t/\tau_0 = 3.33$ and $t/\tau_0 = 6.67$. The presence of relaxation generates perturbation decay. The shorter the relaxation time, the more slowly it moves. It seems that when $\tau/\tau_0 = 200$ the perturbation maximum moves more slowly than when $\tau/\tau_0 = 2000$. However, a reduction in the wave propagation velocity compensates the pressure reduction in the rear part of the wave due to deeper relaxation transmission. In all waves there exists a dilatation wave, the pressure in which approaches gradually the original value as the wave moves away.

The density perturbation evolution in media which are homogeneous for $\omega\tau \simeq 3$ (dashed lines) and periodic for $\omega\tau \simeq 1.5$ (solid lines) is shown in Fig. 3. Though the initial perturbations in them are identical, the initial perturbation of the mean density in the periodic medium is more than two times smaller than in the homogeneous medium. Their qualitative behavior is similar. The propagation velocities of the mid-fronts of the density and pressure perturbations coincide. A phase of reduced density is generated, lagging with respect to the reduced pressure phase.

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