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The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method

V.O. Vakhnenko [a,*], E.J. Parkes [b]

[a] *Institute of Geophysics, Ukrainian Academy of Sciences, 03680 Kiev, Ukraine*
[b] *Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK*

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Abstract

A Bäcklund transformation both in bilinear form and in ordinary form for the transformed Vakhnenko equation is derived. An inverse scattering problem is formulated. The inverse scattering method has a third-order eigenvalue problem. A procedure for finding the exact N-soliton solution of the Vakhnenko equation via the inverse scattering method is described. The procedure is illustrated by considering the cases $N = 1$ and $N = 2$.

1. Introduction

This paper deals with the nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0, \tag{1.1}$$

which was first presented by Vakhnenko in [1] to describe high-frequency waves in a relaxing medium [2]. Hereafter (1.1) is referred to as the Vakhnenko equation (VE).

In [1,3] the travelling-wave solutions of the VE were derived. A remarkable feature of the VE is that it has a soliton solution which has loop-like form, i.e. it is a multi-valued function (see Fig. 1 in [1]). The physical interpretation of the multi-valued functions that describe the loop-like soliton solutions was given in [2]. In [4] the symmetry properties of the VE were studied. It is significant that the loop-like solutions are stable to long-wavelength perturbations [3] and that the introduction of a dissipative term, with dissipation parameter less than some limit value, does not destroy these loop-like solutions [2]. Recently we obtained the two-loop soliton solution to the VE both by use of Hirota's method [5] and by use of elements of the inverse scattering transform (IST) procedure for the KdV equation [6]. We have obtained the N-loop soliton solution to the VE by use of Hirota's method [7]. As we have shown that the VE is integrable, the IST is the most appropriate way of tackling the initial value problem. In order to use the IST method one first has to formulate the associated eigenvalue problem. This can be achieved by finding a Bäcklund transformation associated with the VE; it is well known that the Bäcklund transformation is one of the analytical tools for dealing with soliton problems and has a close relationship to the IST method [8–10]. In this paper we use the IST method to find the exact N-soliton solution of the VE.

In Section 2 we summarise how we previously introduced new independent coordinates in terms of which the solution to the VE is given by single-valued parametric relations [5,6]. The transformation into these coordinates is the key to solving the problem of the interaction of solitons as well as explaining multiple-valued solutions [2]. This

* Corresponding author.
E-mail addresses: vakhnenko@bitp.kiev.ua (V.O. Vakhnenko), ejp@maths.strath.ac.uk (E.J. Parkes).

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transformation also leads to a transformed VE that can be expressed in bilinear form in terms of the Hirota D operator [8]. In Section 3 we present the Bäcklund transformation in bilinear form and in ordinary form for the VE written in terms of the new independent variables. The former type of Bäcklund transformation was first introduced by Hirota [9] and has the advantage that the transformation equations are linear with respect to each dependent variable. The Bäcklund transformation in ordinary form enables one to relate pairs of solutions of the VE. In Section 4 we find that the IST problem for the transformed VE involves a third-order eigenvalue problem. The inverse problem for certain third-order spectral equations has been considered by Kaup [11] and Caudrey [12,13]. In Section 5 we adapt the results obtained by these authors to the present problem and describe a procedure for using the IST to find the N-soliton solution to the transformed VE, and hence to the VE itself. In Section 6 we consider the cases $N = 1$ and $N = 2$ in more detail.

2. Equation in new coordinates

As previously [5,6], we define new independent variables (T, X) by the transformations

$$\varphi\,\mathrm{d}T = \mathrm{d}x - u\,\mathrm{d}t, \quad X = t. \tag{2.1}$$

The function φ is to be obtained. It is an additional dependent variable in equation system (2.3), (2.4) to which we reduce the original equation (1.1). The transformation (2.1) is similar to the transformation between Eulerian coordinates (x, t) and Lagrangian coordinates (T, X). We then require $T = x$ if there is no perturbation, i.e. if $u(x, t) = 0$. Hence $\varphi = 1$ when $u(x, t) = 0$. For example, it may be shown from Eqs. (12) and (14) in [1] that $\varphi = 1 - u/v$ for the one-loop soliton solution.

In terms of the coordinates (T, X) and the unknown $U(X, T) \equiv u(x, t)$, Eq. (1.1) becomes

$$\varphi^{-1}\frac{\partial}{\partial T}\frac{\partial}{\partial X}U + U = 0. \tag{2.2}$$

An equation for the variable φ can be obtained in the following way. Noting that the transformation inverse to (2.1) is

$$\mathrm{d}x = \varphi\,\mathrm{d}T + U\,\mathrm{d}X,$$

and taking into account the condition that $\mathrm{d}x$ is an exact differential, we get

$$\frac{\partial\varphi}{\partial X} = \frac{\partial U}{\partial T}. \tag{2.3}$$

This equation, together with (2.2) rewritten in the form

$$\frac{\partial^2\varphi}{\partial X^2} + U\varphi = 0, \tag{2.4}$$

is the main system of equations. In terms of the coordinates (T, X) the solution is given by single-valued parametric relations. The transformation into these coordinates is the key to solving the problem of the interaction of solitons as well as explaining multiple-valued solutions [2]. The equation system (2.3), (2.4) can be reduced to a nonlinear equation in the unknown W defined by

$$W_X = U \tag{2.5}$$

as follows. As in [5,6], we study solutions U that vanish as $|X| \to \infty$ or, equivalently, solutions for which W tends to a constant as $|X| \to \infty$. From (2.3) and (2.5) and the requirement that $\varphi \to 1$ as $|X| \to \infty$ we have $\varphi = 1 + W_T$; then (2.4) may be written

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{2.6}$$

Furthermore, it then follows that the original independent space coordinate x is given by

$$x = \theta(X, T) := x_0 + T + W(X, T), \tag{2.7}$$

where

$$W = \int_{-\infty}^{X} U(X', T)\,\mathrm{d}X' \tag{2.8}$$

and x_0 is an arbitrary constant. Now the solution to the VE is given in parametric form by

$$u(x,t) = U(t,T), \quad x = \theta(t,T), \tag{2.9}$$

where, for T fixed, the functions $U(t,T)$ and $\theta(t,T)$ are single valued.

Finally, by taking

$$W = 6(\ln f)_X, \tag{2.10}$$

where f is a function of X and T, we observe that (2.6) may be written as the bilinear equation [5]

$$\left(D_T D_X^3 + D_X^2\right) f \cdot f = 0, \tag{2.11}$$

where D is the Hirota binary operator defined by

$$\left(D_T^m D_X^n\right) a \cdot b := \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial T'}\right)^m \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial X'}\right)^n a(X,T) b(X',T')\bigg|_{X'=X,T'=T}$$

for non-negative integers m and n [8].

In passing we note that the Hirota–Satsuma equation (HSE) for shallow water waves [14] may be written in the form

$$W_{XXT} + (1 + W_T) W_X - W_T = 0, \tag{2.12}$$

or in bilinear form as

$$\left(D_T D_X^3 + D_X^2 - D_T D_X\right) f \cdot f = 0. \tag{2.13}$$

Clearly (2.12) and (2.13) are similar to, but cannot be transformed into, (2.6) and (2.11), respectively. Hence solutions to the HSE cannot be transformed into solutions of the transformed VE. The solution to the HSE by Hirota's method is given in [14]; however, as far as we are aware, the solution by the IST method has not been given explicitly in the literature.

Our main aim in this paper is to give the details of the IST method for solving the transformed VE. This is done in Section 5. First, however, we need to formulate the scattering problem. This is done in Section 4 using results from Section 3.

3. Bäcklund transformation for the transformed Vakhnenko equation

In this section we present a Bäcklund transformation for (2.11), the bilinear form of Eq. (2.6).

We follow the method developed in [9]. First we define P as follows:

$$P := 2\left\{\left[\left(D_T D_X^3 + D_X^2\right) f' \cdot f'\right] f f - f' f' \left[\left(D_T D_X^3 + D_X^2\right) f \cdot f\right]\right\}, \tag{3.1}$$

where $f \neq f'$. We aim to find a pair of equations such that each equation is linear in each of the dependent variables f and f', and such that together f and f' satisfy $P = 0$. (It then follows that if f is a solution of (2.11) then so is f' and vice versa.) The pair of equations is the required Bäcklund transformation.

We show that the Bäcklund transformation is given by the two equations

$$\left(D_X^3 - \lambda\right) f' \cdot f = 0, \tag{3.2}$$

$$\left(3 D_X D_T + 1 + \mu D_X\right) f' \cdot f = 0, \tag{3.3}$$

where $\lambda = \lambda(X)$ is an arbitrary function of X and $\mu = \mu(T)$ is an arbitrary function of T.

We prove that together f and f', as determined by Eqs. (3.2) and (3.3), satisfy $P = 0$ as follows. By using the identities (VII.3), (VII.4) from [15], and Eq. (5.86) from [8] we may express P in the following form:

$$\begin{aligned} P = {} & D_T\left[\left(D_X^3 f' \cdot f\right) \cdot (f' f) - 3\left(D_X^2 f' \cdot f\right) \cdot \left(D_X f' \cdot f\right)\right] + D_X\left[3\left(D_T D_X^2 f' \cdot f\right) \cdot (f' f) - 6\left(D_X D_T f' \cdot f\right) \cdot \left(D_X f' \cdot f\right)\right. \\ & \left. - 3\left(D_X^2 f' \cdot f\right) \cdot \left(D_T f' \cdot f\right) + 4\left(D_X f' \cdot f\right) \cdot (f' f)\right]. \end{aligned} \tag{3.4}$$

On using the identities (A.1) and (A.2) given in Appendix A we can rewrite P in the following form:

$$P = 4 D_T\left(\left\{D_X^3 - \lambda(X)\right\} f' \cdot f\right) \cdot (f' f) - 4 D_X\left(\left\{3 D_T D_X + 1 + \mu(T) D_X\right\} f' \cdot f\right) \cdot \left(D_X f' \cdot f\right). \tag{3.5}$$

From (3.5) it follows that if Eqs. (3.2) and (3.3) hold then $P = 0$ as required.

Thus we have proved that the pair of equations (3.2) and (3.3) constitute a Bäcklund transformation in bilinear form for Eq. (2.11). Separately these two equations appear as part of the Bäcklund transformation for other nonlinear evolution equations. For example,Eq. (3.2) is the same as one of the equations that is part of the Bäcklund transformation for a higher-order KdV equation (see Eq. (5.139) in [8]), and Eq. (3.3) is similar to (5.132) in [8] that is part of the Bäcklund transformation for a model equation for shallow water waves.

The inclusion of μ in the operator $3D_T + \mu(T)$ which appears in (3.5) corresponds to a multiplication of f and f' by terms of the form $e^{g(T)}$ and $e^{g'(T)}$, respectively; from (2.10) we see that this has no effect on W or W'. Hence, without loss of generality, we may take $\mu = 0$ in (3.3) if we wish.

Following the procedure given in [8,16], we can rewrite the Bäcklund transformation in ordinary form in terms of the potential W given by (2.8). In new variables defined by

$$\phi = \ln f'/f, \qquad \rho = \ln f'f, \tag{3.6}$$

Eqs. (3.2) and (3.3) have the form

$$\phi_{XXX} + 3\phi_X \rho_{XX} + \phi_X^3 - \lambda = 0, \tag{3.7}$$

$$3(\rho_{XT} + \phi_X \phi_T) + 1 + \mu\phi_X = 0, \tag{3.8}$$

respectively, where we have used results similar to (XI.1)–(XI.3) in [8]. From the definitions (2.10) and (3.6), different solutions W, W' of Eq. (2.6) are related to ϕ and ρ by

$$W' - W = 6\phi_X, \qquad W' + W = 6\rho_X. \tag{3.9}$$

Substitution of (3.9) into (3.7) and (3.8) with $\mu = 0$ leads to

$$(W' - W)_{XX} + \frac{1}{2}(W' - W)(W' + W)_X + \frac{1}{36}(W' - W)^3 - 6\lambda = 0, \tag{3.10}$$

$$(W' - W)\left[3(W' + W)_{XT} + \frac{1}{2}(W' - W)(W' - W)_T\right] - 6(W' - W)_X\left[1 + \frac{1}{2}(W' + W)_T\right] = 0, \tag{3.11}$$

respectively. The required Bäcklund transformation in ordinary form is the pair of equations (3.10) and (3.11).

4. Formulation of the inverse scattering eigenvalue problem

In this section we will show that the IST problem for the transformed VE in the form (2.6) has a third-order eigenvalue problem that is similar to the one associated with a higher-order KdV equation [11,16], a Boussinesq equation [11,12,17,18], and a model equation for shallow water waves [8,15].

Introducing the function

$$\psi = f'/f, \tag{4.1}$$

and taking into account Eqs. (2.5) and (2.10), we find that Eqs. (3.2) and (3.3) reduce to

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{4.2}$$

$$3\psi_{XT} + (W_T + 1)\psi + \mu\psi_X = 0, \tag{4.3}$$

respectively, where we have used results similar to (X.1)–(X.3) in [8]. It may be shown from (4.2) and (4.3) that, even with $\mu \neq 0$,

$$[W_{XXT} + (1 + W_T)W_X]_X\psi + 3\lambda_X\psi_T = 0.$$

Hence (2.6) is the condition for $\lambda_X = 0$, and hence for λ to be constant. Constant λ is what is required in the IST problem.

Since (4.2) and (4.3) are alternative forms of Eqs. (3.2) and (3.3), respectively, it follows that the pair of equations (4.2) and (4.3) is associated with the transformed VE (2.6) considered here. Thus the IST problem is directly related to a spectral equation of third order, namely (4.2). The inverse problem for certain third-order spectral equations has been considered by Kaup [11] and Caudrey [12,13]. As expected (4.2) and (4.3) are similar to, but cannot be transformed into, the corresponding equations for the HSE (see Eqs. (A8a) and (A8b) in [19]). Clarkson and Mansfield [20] note that the scattering problem for the HSE is similar to that for the Boussinesq equation which has been studied comprehensively by Deift et al. [18].

5. The N-soliton solution

The general theory of the inverse scattering problem for N spectral equations has been developed in [12]. Following the procedure given in [12], the spectral equation (4.2) can be rewritten in the form

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X,\zeta)] \cdot \boldsymbol{\psi} \tag{5.1}$$

by putting

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -U & 0 \end{pmatrix}. \tag{5.2}$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_j(\zeta)$ and left- and right-eigenvectors $\tilde{\boldsymbol{v}}_j(\zeta)$ and $\boldsymbol{v}_j(\zeta)$, respectively, where

$$\lambda_j(\zeta) = \omega_j\zeta, \qquad \lambda_j^3(\zeta) = \lambda, \qquad \boldsymbol{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \end{pmatrix}, \qquad \tilde{\boldsymbol{v}}_j(\zeta) = \begin{pmatrix} \lambda_j^2, & \lambda_j, & 1 \end{pmatrix}, \tag{5.3}$$

and $\omega_j = \mathrm{e}^{\mathrm{i}2\pi(j-1)/3}$ are the cube roots of 1.

The solution of the linear equation (2.6), or equivalently Eq. (5.1), has been obtained by Caudrey [12] in terms of Jost functions $\boldsymbol{\phi}_j(X,\zeta)$ which have the asymptotic behaviour

$$\boldsymbol{\Phi}_j(X,\zeta) := \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X,\zeta) \to \boldsymbol{v}_j(\zeta) \quad \text{as } X \to -\infty. \tag{5.4}$$

Here T is regarded as a parameter; the T-evolution of the scattering data will be taken into account later. The solution of the direct problem is given by the equation system (4.5) in [12]. We shall restrict our attention to the N-soliton solution. To do this we consider equation (6.20) from [12] by putting $Q_{ij}(\zeta) \equiv 0$. Then there is only the bound state spectrum which is associated with the soliton solutions.

Let the bound state spectrum be defined by K poles. The relation (4.5) from [12] is reduced to the form

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)}\Phi_1(X,\omega_j\zeta_1^{(k)}). \tag{5.5}$$

We need only consider the function $\Phi_1(X,\zeta)$ since there is a set of symmetry properties as for the Boussinesq equation, namely the properties (6.15) in [12] for Jost functions $\boldsymbol{\phi}_j(X,\zeta)$. Eq. (5.5) involves the spectral data, namely the poles $\zeta_i^{(k)}$ and the quantities $\gamma_{ij}^{(k)}$. First we will prove that $\operatorname{Re}\lambda = 0$ for compact support. From Eq. (4.2) we have

$$(\psi_X)_{XXX} + (U\psi_X)_X - \lambda\psi_X = 0 \tag{5.6}$$

and together with Eq. (4.2) this enables us to write

$$\frac{\partial}{\partial X}\left(\frac{\partial^2}{\partial X^2}\psi_X\psi^* - 3\psi_{XX}\psi_X^* + U\psi_X\psi^*\right) - 2\operatorname{Re}\lambda\psi_X\psi^* = 0. \tag{5.7}$$

Integrating Eq. (5.7) over all values of X, we obtain that for compact support $\operatorname{Re}\lambda = 0$ since, in the general case, $\int_{-\infty}^{\infty}\psi_X\psi^*\,\mathrm{d}X \neq 0$.

As follows from Eqs. (2.12), (2.13), (2.36) and (2.37) of [11], $\psi_X(\zeta)$ is related to the adjoint states $\psi^A(-\zeta)$. In the usual manner, using the adjoint states and Eq. (14) from [13], and Eq. (2.37) from [11], one can obtain

$$\phi_{1X}(X,\zeta) = \frac{\mathrm{i}}{\sqrt{3}}[\phi_{1X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) - \phi_{1X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)]. \tag{5.8}$$

It is easily seen that if $\zeta_1^{(1)}$ is a pole of $\phi_1(X,\zeta)$, then there is a pole either at $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_2\zeta)$ has a pole), or at $\zeta_1^{(2)} = -\omega_3\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_3\zeta)$ has a pole). For definiteness let $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$. Then, as follows from (5.8), $-\omega_3\zeta_1^{(2)}$ should be a pole. However, this pole coincides with the pole $\zeta_1^{(1)}$, since $-\omega_3\zeta_1^{(2)} = -\omega_3(-\omega_2)\zeta_1^{(1)} = \zeta_1^{(1)}$. Hence, the poles appear in pairs, $\zeta_1^{(2n-1)}$ and $\zeta_1^{(2n)}$, under the condition $\zeta_1^{(2n)}/\zeta_1^{(2n-1)} = -\omega_2$, where n is the pair number.

Let us consider N pairs of poles, i.e. in all there are $K = 2N$ poles over which the sum is taken in (5.5). For the pair n ($n = 1, 2, \ldots, N$) we have the properties

$$\text{(i)}\quad \zeta_1^{(2n-1)} = \mathrm{i}\omega_2\xi_n, \qquad \text{(ii)}\quad \zeta_1^{(2n)} = -\mathrm{i}\omega_3\xi_n. \tag{5.9}$$

Since U is real and λ is imaginary, ξ_k is real. The relationships (5.9) are in line with the condition (2.33) from [11]. These relationships are also similar to Eqs. (6.24) and (6.25) in [12], while $\gamma_{1j}^{(k)}$ turns out to be different from $\tilde{\gamma}_{1j}^{(k)}$ for the Boussinesq equation (see Eqs. (6.24) and (6.25) in [12]). Indeed, by considering (5.8) in the vicinity of the first pole $\zeta_1^{(2n-1)}$ of the pair n and using the relation (5.5), one can obtain a relation between $\gamma_{12}^{(2n-1)}$ and $\gamma_{13}^{(2n)}$. In this case the functions $\phi_{1X}(X,\zeta)$, $\phi_1(X,-\omega_2\zeta)$, $\phi_{1X}(X,-\omega_2\zeta)$ also have poles here, while the functions $\phi_1(X,-\omega_3\zeta)$, $\phi_{1X}(X,-\omega_3\zeta)$ do not have poles here. Substituting $\phi_1(X,\zeta)$ in the form (5.4), (5.5) into Eq. (5.8) and letting $X\to-\infty$, we have the ratio $\gamma_{13}^{(2n)}/\gamma_{12}^{(2n-1)} = \omega_2$ and $\gamma_{12}^{(2n)} = \gamma_{13}^{(2n-1)} = 0$. Therefore the properties of $\gamma_{ij}^{(k)}$ should be defined by the relationships

$$\begin{aligned}&\text{(i)}\quad \gamma_{12}^{(2n-1)} = \omega_2\beta_k, \qquad \gamma_{13}^{(2n-1)} = 0,\\ &\text{(ii)}\quad \gamma_{12}^{(2n)} = 0, \qquad \gamma_{13}^{(2n)} = \omega_3\beta_k,\end{aligned} \tag{5.10}$$

where, as will be proved below, β_k is real when U is real.

By expanding $\Phi_1(X,\zeta)$ as an asymptotic series in $\lambda_1^{-1}(\zeta)$, one can obtain (cf. Eq. (2.7) in [11])

$$\Phi_1(X,\zeta) = 1 - \frac{1}{3\lambda_1(\zeta)}[W(X) - W(-\infty)] + \mathrm{O}(\lambda_1^{-2}(\zeta)). \tag{5.11}$$

On the other hand, by defining

$$\Psi_k(X) = \sum_{j=2}^{3}\gamma_{1j}^{(k)}\exp\{\lambda_j(\zeta_1^{(k)})X\}\Phi_1(X,\omega_j\zeta_1^{(k)}), \tag{5.12}$$

we may rewrite the relationship (5.5) as (see, for instance, Eqs. (6.33) and (6.34) in [12])

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{2N}\frac{\exp\{-\lambda_1(\zeta_1^{(k)})X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)}\Psi_k(X). \tag{5.13}$$

From (5.11) and (5.13) it may be shown that (cf. Eq. (6.38) in [12])

$$W(X) - W(-\infty) = -3\sum_{k=1}^{2N}\exp\{-\lambda_1(\zeta_1^{(k)})X\}\Psi_k(X) = 3\frac{\partial}{\partial X}\ln(\det M). \tag{5.14}$$

The matrix M is defined as in the relationship (6.36) in [12] by

$$M_{kl}(X) = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})]X\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})}. \tag{5.15}$$

Now let us consider the T-evolution of the spectral data. By analyzing the solution of Eq. (4.3) when $X\to-\infty$, we find that $\phi_i(X,T,\zeta) = \exp[-(3\lambda_i(\zeta))^{-1}T]\phi_i(X,0,\zeta)$. Hence the T-evolution of the scattering data is given by the relationships (with $k = 1,2,\ldots,K$)

$$\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0), \qquad \gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0)\exp\{[-(3\lambda_j(\zeta_1^{(k)}))^{-1} + (3\lambda_1(\zeta_1^{(k)}))^{-1}]T\}. \tag{5.16}$$

The final result, including the T-evolution, for the N-soliton solution of the transformed VE is

$$U(X,T) = 3\frac{\partial^2}{\partial X^2}\ln(\det M(X,T)), \tag{5.17}$$

where M is the $2N\times 2N$ matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}(0)\frac{\exp\{[-(3\lambda_j(\zeta_1^{(k)}))^{-1} + (3\lambda_1(\zeta_1^{(k)}))^{-1}]T + (\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)}))X\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})}, \tag{5.18}$$

and

$$\begin{aligned}&n = 1,2,\ldots,N, \quad m = 2n-1,\\ &\lambda_1(\zeta_1^{(m)}) = \mathrm{i}\omega_2\xi_m, \quad \lambda_2(\zeta_1^{(m)}) = \mathrm{i}\omega_3\xi_m, \quad \gamma_{12}^{(m)}(0) = \omega_2\beta_m, \quad \gamma_{13}^{(m)}(0) = 0,\end{aligned}$$

$$\lambda_1(\zeta_1^{(m+1)}) = -\mathrm{i}\omega_3\xi_m, \quad \lambda_3(\zeta_1^{(m+1)}) = -\mathrm{i}\omega_2\xi_m, \quad \gamma_{12}^{(m+1)}(0) = 0, \quad \gamma_{13}^{(m+1)}(0) = \omega_3\beta_m.$$

For the N-soliton solution there are N arbitrary constants ξ_m and N arbitrary constants β_m.

6. Examples of one- and two-soliton solutions

In order to obtain the one-soliton solution of the transformed VE (2.6) we need first to calculate the 2×2 matrix M according to (5.18) with $N = 1$. We find that the matrix is

$$\begin{pmatrix} 1 - \frac{\omega_2\beta_1}{\sqrt{3}\xi_1}\exp[\sqrt{3}\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] & \frac{\mathrm{i}\omega_3\beta_1}{2\xi_1}\exp[2\mathrm{i}\omega_3\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] \\ \frac{-\mathrm{i}\omega_2\beta_1}{2\xi_1}\exp[-2\mathrm{i}\omega_2\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] & 1 - \frac{\omega_3\beta_1}{\sqrt{3}\xi_1}\exp[\sqrt{3}\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] \end{pmatrix}, \tag{6.1}$$

and its determinant is

$$\det M = \left\{1 + \frac{\beta_1}{2\sqrt{3}\xi_1}\exp\left[\sqrt{3}\xi_1\left(X - \frac{T}{3\xi_1^2}\right)\right]\right\}^2. \tag{6.2}$$

Consequently, from (5.17), the one-soliton solution of the transformed VE as obtained by the IST method is

$$U = \frac{9}{2}\xi_1^2\,\mathrm{sech}^2\left[\frac{\sqrt{3}}{2}\xi_1\left(X - \frac{T}{3\xi_1^2}\right) + \alpha_1\right], \tag{6.3}$$

here $\alpha_1 = \frac{1}{2}\ln(\beta_1/2\sqrt{3}\xi_1)$ is an arbitrary constant. Since U is real, it follows from (6.3) that α_1 is real, and so β_1 is also real. By writing $\sqrt{3}\xi_1/2 = k$ in (6.3) we recover the one-soliton solution as we found previously by Hirota's method (see Eq. (3.4) in [5]).

It is of interest to compare Eq. (6.3) with the solution of the fifth-order KdV-like equation discussed in [11]. The spectral equation (4.2) is the same as that given by (1.1) (with $R = 0$) in [11], whereas the equation that governs the time dependence of ψ, i.e. (4.3), is different from (1.2) in [11]. Thus the X dependence of (6.3) should agree with the x dependence of the solution given by (3.30) in [11]. With the identification $U = 6Q$, $\xi_1 = \eta$, this is indeed the case.

Let us now consider the two-soliton solution of the transformed VE. In this case M is a 4×4 matrix. We will not give the explicit form here, but we find that

$$\det M = \left(1 + q_1^2 + q_2^2 + b^2 q_1^2 q_2^2\right)^2, \tag{6.4}$$

where

$$q_i = \exp\left[\frac{\sqrt{3}}{2}\xi_i\left(X - \frac{T}{3\xi_i^2}\right) + \alpha_i\right], \quad b^2 = \left(\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1}\right)^2 \frac{\xi_1^2 + \xi_2^2 - \xi_1\xi_2}{\xi_1^2 + \xi_2^2 + \xi_1\xi_2}, \tag{6.5}$$

and $\alpha_i = \frac{1}{2}\ln(\beta_i/2\sqrt{3}\xi_i)$ are arbitrary constants. The two-soliton solution to the transformed VE as found by the IST method is given by (5.17) together with (6.4). With the identification $\sqrt{3}\xi_i/2 = k_i$ $(i = 1, 2)$ we recover the two-soliton solution as given by Hirota's method (see Eqs. (4.1)–(4.5) in [5]).

Finally we note that comparison of (5.17) with (2.10) together with (2.5) shows that

$$\ln(\det M) = 2\ln f \tag{6.6}$$

so that $\det M$ is a perfect square for arbitrary N.

7. Conclusion

We have found the Bäcklund transformation both in bilinear form and in ordinary form for the transformed VE. It enabled us to formulate an IST problem for the transformed VE which is directly related to a spectral equation of third order. We have described how to obtain the N-soliton solution to the VE.

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Appendix A

The following identities (A.1), (A.2) are required in Section 3:

$$D_X^3(D_Tf'\cdot f)\cdot(ff') = D_T\left[(D_X^3f'\cdot f)\cdot(ff') - 3(D_X^2f'\cdot f)\cdot(D_Xf'\cdot f)\right], \tag{A.1}$$

$$\begin{aligned}4D_T(D_X^2f'\cdot f)\cdot(D_Xf'\cdot f) = {} & D_X[(D_TD_X^2f'\cdot f)\cdot(f'f) + 2(D_TD_Xf'\cdot f)\cdot(D_Xf'\cdot f) - (D_X^2f'\cdot f)\cdot(D_Tf'\cdot f)] \\ & - D_X^3(D_Tf'\cdot f)\cdot(f'f).\end{aligned} \tag{A.2}$$

Identities (A.1) and (A.2) come from

$$\begin{aligned}&\exp(D_1)[\exp(D_2)f'\cdot f]\cdot[\exp(D_3)f'\cdot f] \\ &\quad = \exp\left(\frac{1}{2}\{D_2 - D_3\}\right)\left[\exp\left\{\frac{1}{2}(D_2+D_3)+D_1\right\}f'\cdot f\right]\cdot\left[\exp\left\{\frac{1}{2}(D_2+D_3)-D_1\right\}f'\cdot f\right]\end{aligned} \tag{A.3}$$

which is Eq. (5.83) in [8], where $D_i := \varepsilon_i D_X + \delta_i D_T$. In the order $\varepsilon_1^3\delta_3$ (A.3) yields (A.1), and in the order $\delta_1\varepsilon_2^2\varepsilon_3$ (A.3) yields (A.2).

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