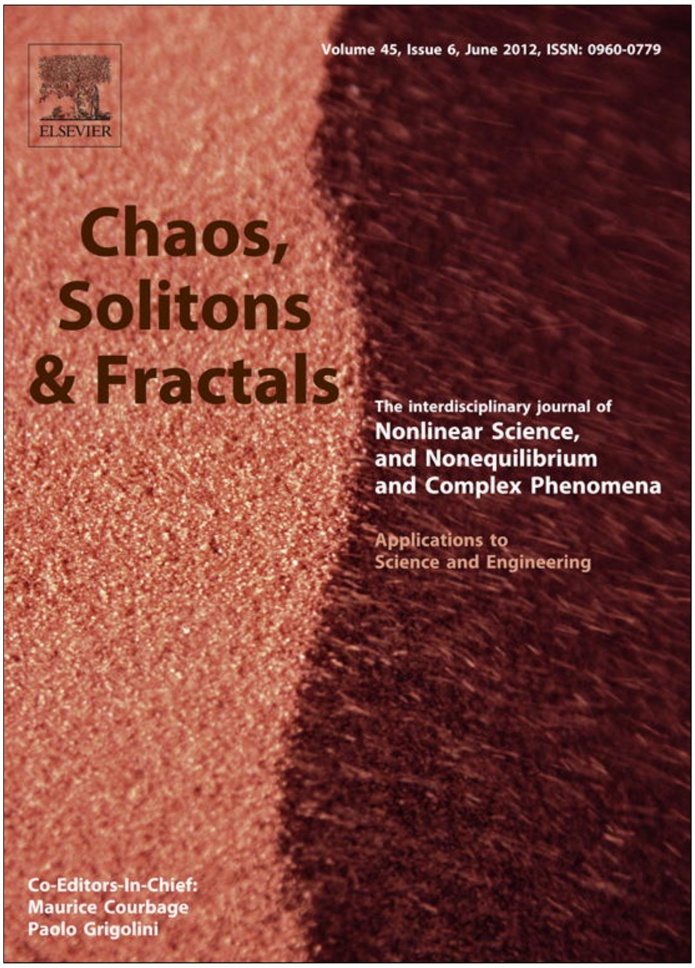
Volume 45, Issue 6, June 2012, ISSN: 0960-0779
ELSEVIER
Chaos, Solitons & Fractals
The interdisciplinary journal of
Nonlinear Science, and Nonequilibrium and Complex Phenomena
Applications to Science and Engineering
Co-Editors-In-Chief:
Maurice Courbage
Paolo Grigolini

Chaos, Solitons & Fractals 45 (2012) 846–852

Contents lists available at SciVerse ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

The singular solutions of a nonlinear evolution equation taking continuous part of the spectral data into account in inverse scattering method

V.O. Vakhnenko [a,*], E.J. Parkes [b]

[a] Institute of Geophysics, National Academy of Sciences of Ukraine, 01054 Kyïv, Ukraine
[b] Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, United Kingdom

ARTICLE INFO

Article history:
Received 29 May 2011
Accepted 29 February 2012
Available online 5 April 2012

ABSTRACT

A procedure for finding the solutions of the Vakhnenko–Parkes equation by means of the inverse scattering method is described. Both the bound state spectrum and the continuous spectrum are considered in the associated eigenvalue problem. The suggested special form of the singularity function gives rise to periodic solutions. The interaction of a soliton with a one-mode periodic wave is studied.

1. Introduction

In many applications of physics and technology it is significant to look for exact solutions of nonlinear evolution equations. Various effective approaches have been developed to construct exact wave solutions of completely integrable equations. One of the fundamental direct methods is undoubtedly the Hirota bilinear method [1,2] which possesses significant features that make it practical for the determination of multiple-soliton solutions. However, the direct methods can be applied only for finding the solitary wave solutions or the traveling-wave solutions. In this sense, the inverse scattering method is the most appropriate way of tackling the initial value problem although its employment is a fairly difficult procedure [3–5].

In this paper we will consider the nonlinear evolution equation:

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{1.1}$$

This equation arises from the Vakhnenko equation [6–8]:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0, \tag{1.2}$$

through the transformation:

$$\begin{aligned} &u(x,t) := U(X,T) = W_X(X,T),\\ &x := x_0 + T + W(X,T), \quad t := X. \end{aligned} \tag{1.3}$$

The details of the transformation (1.3) can be found in [9,10]. The corresponding governing equation for U, namely:

$$UU_{XXT} - U_X U_{XT} + U^2 U_T = 0, \tag{1.4}$$

is given in [9]. Following the papers [11–13], hereafter (1.1) (or equivalently (1.4)) is referred to as the Vakhnenko–Parkes equation (VPE).

Recently the Hirota method [2,9,10] as well as the inverse scattering method [14] have been applied to obtain the exact N-soliton solutions of the VPE. In this paper we use the inverse scattering transform method to study additionally the periodic solutions of the VPE (1.1) associated with the continuum part of the spectral data as well as to investigate the interaction of a soliton with a periodic wave.

In Section 2 we formulate the spectral problem for the VPE by adapting the results given by Caudrey [15] and by Kaup [16]. For convenience, in Section 3 we present briefly

* Corresponding author.
E-mail addresses: vakhnenko@ukr.net (V.O. Vakhnenko), e.j.parkes@strath.ac.uk (E.J. Parkes).
URLs: http://www.igph.kiev.ua/vakhnenko/vakhnenko.html (V.O. Vakhnenko), http://personal.strath.ac.uk/e.j.parkes/ (E.J. Parkes).

0960-0779/$ - see front matter
http://dx.doi.org/10.1016/j.chaos.2012.02.019

the results that we obtained in [14] for the N-soliton solution corresponding to the bound state spectrum. In Section 4 we develop the corresponding results for the continuous spectrum. As a particular example, we find the one-mode solution. We then investigate how this solution interacts with a soliton solution. Our results are summarized in Section 5.

2. The spectral problem for the VPE

In order to use the inverse scattering method, one first has to formulate the associated eigenvalue problem. In [14] it is shown that the pair of equations

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{2.1}$$

$$3\psi_{XT} + (W_T + 1)\psi = 0 \tag{2.2}$$

is associated with the VPE (1.1) considered here. Note that the inverse scattering transform problem is related to a spectral equation of third order (2.1). The inverse problem for third-order spectral equations has been considered by Caudrey [15] and Kaup [16]. We adapt the results obtained by these authors to the present spectral problem and describe a procedure for using the inverse scattering transform method to find the solutions of the VPE.

We use the general theory of the inverse scattering problem for N spectral equations which has been developed by Caudrey [15]. According to [15] the spectral Eq. (2.1) can be rewritten in the form:

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X,\zeta)]\cdot\boldsymbol{\psi} \tag{2.3}$$

with

$$\boldsymbol{\psi} = \begin{pmatrix}\psi\\ \psi_X\\ \psi_{XX}\end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix}0&1&0\\0&0&1\\ \lambda&0&0\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix}0&0&0\\0&0&0\\0&-W_X&0\end{pmatrix}. \tag{2.4}$$

The matrix $\mathbf{A}$ has the eigenvalues $\lambda_j(\zeta)$ and left- and right-eigenvectors $\tilde{\boldsymbol{v}}_j(\zeta)$ and $\boldsymbol{v}_j(\zeta)$ respectively ($j = 1,2,3$). In the case considered here we define:

$$\lambda_j(\zeta) = \omega_j\zeta, \quad \lambda_j^3(\zeta) = \lambda, \quad \boldsymbol{v}_j(\zeta) = \begin{pmatrix}1\\ \lambda_j\\ \lambda_j^2\end{pmatrix},$$
$$\tilde{\boldsymbol{v}}_j(\zeta) = \begin{pmatrix}\lambda_j^2 & \lambda_j & 1\end{pmatrix}, \tag{2.5}$$

where $\omega_j = e^{2\pi i(j-1)/3}$ are the cube of roots of 1.

The solution of the linear Eq. (2.1), or equivalently Eq. (2.3), has been obtained by Caudrey [15] in terms of Jost functions $\boldsymbol{\phi}_j(X,\zeta)$ which have the asymptotic behaviour:

$$\boldsymbol{\Phi}_j(X,\zeta) := \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X,\zeta) \to \boldsymbol{v}_j(\zeta) \quad \text{as} \quad X \to -\infty. \tag{2.6}$$

Here T is regarded as a parameter; the T-evolution of the scattering data will be taken into account later. The solution of the direct problem (2.3) is given by the equation system (4.5) in [15]. Since there is a set of symmetry properties $\phi_1(X,\zeta/\omega_1) = \phi_2(X,\zeta/\omega_2) = \phi_3(X,\zeta/\omega_3)$ (see (6.14) and (6.15) in [15], for example) for Jost functions $\boldsymbol{\phi}_j(X,\zeta)$, we need only consider the element $\phi_1(X,\zeta)$ (as well as $\Phi_1(X,\zeta)$). In the general case it is necessary to take into account both the bound state spectrum and the continuous spectrum. According to the relation (6.20) in [15], the solution of (2.3) is as follows:

$$\begin{aligned}\Phi_1(X,\zeta) = 1 &- \sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)} \\ &\times \frac{\exp\left\{\left[\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(k)}\right)\right]X\right\}}{\lambda_1\left(\zeta_1^{(k)}\right) - \lambda_1(\zeta)}\,\Phi_1\left(X,\omega_j\zeta_1^{(k)}\right) \\ &+ \frac{1}{2\pi i}\int\sum_{j=2}^{3}Q_{1j}(\zeta') \\ &\times \frac{\exp\{[\lambda_j(\zeta') - \lambda_1(\zeta')]X\}}{\zeta' - \zeta}\;\Phi_1^{\pm}(X,\omega_j\zeta')d\zeta'.\end{aligned} \tag{2.7}$$

Eq. (2.7) contains the spectral data, namely K poles with the quantities $\gamma_{1j}^{(k)}$ for the bound state spectrum as well as the functions $Q_{1j}(\zeta')$ given along all the boundaries of regular regions for the continuous spectrum. The boundaries between regions, where the Jost functions $\boldsymbol{\phi}_1(X,\zeta)$ is regular, appear at $Re(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ over all $j \neq 1$ [15]. The singularities on boundaries of these regions within the complex ζ-plane are taken into account by the third term in the relation (2.7). The integral in (2.7) is along all the boundaries (see the dashed lines in Fig. 1).

3. The standard and singular soliton solutions

For convenience, in this section we repeat the results from [14] for the N-soliton solution. The bound state spectrum is associated with soliton solutions; for this case we put $Q_{1j}(\zeta) \equiv 0$ in (2.7). The poles appear only in pairs. Let

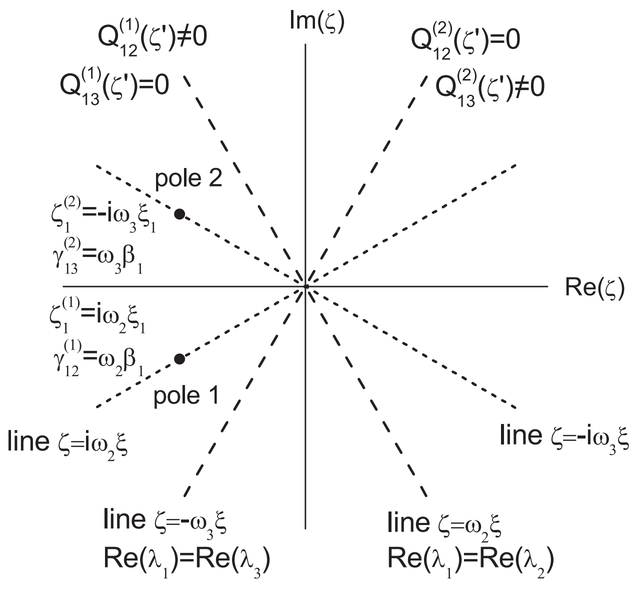

Fig. 1. The regular regions for Jost functions $\phi_1(X,\zeta)$ in the complex ζ-plane. The dashed lines with singularity functions $Q_{1j}(\zeta')$ determine the boundaries between regular regions. The dotted lines are the lines where the poles appear.

there be N pairs, then $K = 2N$ over which the sum is taken in (2.7). In [14] it is proved that for the pair n ($n = 1,2,\ldots,N$) there are the properties

$$\text{(i)}\quad \zeta_1^{(2n-1)} = \mathrm{i}\omega_2\xi_n,\quad \gamma_{12}^{(2n-1)} = \omega_2\beta_n,\quad \gamma_{13}^{(2n-1)} = 0,$$
$$\text{(ii)}\quad \zeta_1^{(2n)} = -\mathrm{i}\omega_3\xi_n,\quad \gamma_{12}^{(2n)} = 0,\quad \gamma_{13}^{(2n)} = \omega_3\beta_n, \tag{3.1}$$

where the constants ξ_n are real always, while the constants β_n can be regarded as real, if we consider only soliton solutions of the VPE.

The N-soliton solution of the VPE is:

$$U(X,T) = W_X(X,T) = 3\frac{\partial^2}{\partial X^2}\ln(\det M(X,T)), \tag{3.2}$$

where M is the $2N \times 2N$ matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}(0) \times \frac{\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T + \left(\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)\right)X\right\}}{\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)}, \tag{3.3}$$

and

$$n = 1,2,\ldots,N,\quad m = 2n-1,$$
$$\lambda_1\left(\zeta_1^{(m)}\right) = i\omega_2\xi_m,\quad \lambda_2\left(\zeta_1^{(m)}\right) = i\omega_3\xi_m,$$
$$\gamma_{12}^{(m)} = \omega_2\beta_m,\quad \gamma_{13}^{(m)} = 0,$$
$$\lambda_1\left(\zeta_1^{(m+1)}\right) = -i\omega_3\xi_m,\quad \lambda_3\left(\zeta_1^{(m+1)}\right) = -i\omega_2\xi_m,$$
$$\gamma_{12}^{(m+1)} = 0,\quad \gamma_{13}^{(m+1)} = \omega_3\beta_m. \tag{3.4}$$

For the N-soliton solution there are N arbitrary real constants ξ_m and N arbitrary constants β_m.

We present the first three results of the calculation of the matrix determinants (3.3). For the sake of convenience we will use the auxiliary function $F(X,T)$ given by the definition $F(X,T) = \sqrt{\det M(X,T)}$. In particular, from (3.3).

(1) for $N = 1$ we have:

$$F = 1 + c_1q_1, \tag{3.5}$$

(2) for $N = 2$ we have:

$$F = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2, \tag{3.6}$$

(3) for $N = 3$ we have:

$$F = 1 + c_1q_1 + c_2q_2 + c_3q_3 + b_{12}c_1c_2q_1q_2 + b_{13}c_1c_3q_1q_3 + b_{23}c_2c_3q_2q_3 + b_{12}b_{13}b_{23}c_1c_2c_3q_1q_2q_3. \tag{3.7}$$

For $N > 3$, the explicit expression for the function $F(X,T)$ can be obtained in a similar manner. The formulas (3.5)–(3.7) contain the quantities b_{ij} that prove that the solitons interact only in pairs. For soliton solutions the quantities c_i, q_i, b_{ij} involved in the above formulas (3.5)–(3.7) have the forms:

$$q_i = \exp\left(\sqrt{3}\xi_iX - (\sqrt{3}\xi_i)^{-1}T\right),\quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i},$$
$$b_{ij} = \left(\frac{\xi_i - \xi_j}{\xi_i + \xi_j}\right)^2\frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}. \tag{3.8}$$

With the above representation of the auxiliary function $F(X,T)$, and taking into account the key relationship (3.2), the explicit solution to the basic nonlinear evolution Eq. (1.1) can be written in the following concise form:

$$U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(F(X,T)). \tag{3.9}$$

The auxiliary function F is complex-valued, in the general case, because the values of β_i (and hence c_i) are complex constants. Special attention must be given to the constants c_i. Since we are interested only in real solutions for W_X with real constants ξ_i, we need restrictions on the constants c_i in (3.8).

In Appendix A we prove that only with real constants c_i do the soliton solutions reduce to real functions. Hence, the restrictions (A.12) are the conditions for real soliton solutions of the VPE.

It should be noted that for different signs of c_i, namely $\alpha_i = c_i/|c_i|$, the soliton solutions can be represented in forms either of standard solitons with $\alpha_i = 1$ or of singular solitons with $\alpha_i = -1$ (see [2]). Additionally to the cases considered by Wazwaz [2], when either all $\alpha_i = 1$ or all $\alpha_i = -1$, the solutions obtained in the form (3.2) and (3.3) allow us to choose the signs for the constants $\alpha_i = \pm 1$ independent of each other.

For example, at $N = 1$ we have either one standard soliton:

$$W = 3\sqrt{3}\xi_1\tanh(\theta_1) \tag{3.10}$$

or one singular soliton:

$$W = 3\sqrt{3}\xi_1\coth(\theta_1) \tag{3.11}$$

with

$$2\theta_1 = \sqrt{3}\xi_1(X - X_1) - (\sqrt{3}\xi_1)^{-1}T,$$
$$X_1 = -\frac{1}{\sqrt{3}\xi_1}\ln|c_1|. \tag{3.12}$$

4. The solutions associated with the continuum part of the spectral data

Now in addition to the bound state spectrum we study the continuous spectrum of the associated eigenvalue problem, i.e. we assume that at least some of the functions $Q_{1j}(\zeta')$ are nonzero. At each fixed $j \neq 1$ the functions $Q_{1j}(\zeta')$ characterize the singularity of $\Phi_1(X,\zeta)$. This singularity can appear only on boundaries between the regular regions on the ζ-plane. The condition $\mathrm{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$ constitutes these boundaries [15]. According to [15] we find that for $\Phi_1(X,\zeta)$ the complex ζ-plane is divided into four regions by two lines

$$\text{(i)}\quad \zeta' = \omega_2\xi,\quad \text{with}\quad Q_{12}^{(1)}(\zeta') \neq 0,\quad Q_{13}^{(1)}(\zeta') \equiv 0,$$
$$\text{(ii)}\quad \zeta' = -\omega_3\xi,\quad \text{with}\quad Q_{12}^{(2)}(\zeta') \equiv 0,\quad Q_{13}^{(2)}(\zeta') \neq 0, \tag{4.1}$$

where ξ is real (see Fig. 1). Analysis shows that the direction of the integration in (2.7) is such that ξ sweeps from $-\infty$ to $+\infty$.

Let us consider the singularity functions $Q_{1j}(\zeta')$ on the boundaries, on which the Jost function $\phi_1(X,\zeta)$ is singular, in the form:

$$\left.\begin{aligned} Q_{12}^{(1)}(\zeta') &= -2\pi i q_{12}^{(1)}\delta(\zeta'-\zeta_1') \\ Q_{13}^{(1)}(\zeta') &= -2\pi i q_{13}^{(1)}\delta(\zeta'-\zeta_1') \equiv 0 \end{aligned}\right\} \text{ on the line } \zeta' = \omega_2\xi,$$
$$\left.\begin{aligned} Q_{12}^{(2)}(\zeta') &= -2\pi i q_{12}^{(2)}\delta(\zeta'-\zeta_2') \equiv 0 \\ Q_{13}^{(2)}(\zeta') &= -2\pi i q_{13}^{(2)}\delta(\zeta'-\zeta_2') \end{aligned}\right\} \text{ on the line } \zeta' = -\omega_3\xi. \tag{4.2}$$

For singularity functions (4.2) and for one pair of poles, the relationship (2.7) is reduced to the form:

$$\begin{aligned} \Phi_1(X,\zeta) = 1 &- \sum_{k=1}^{2}\sum_{j=2}^{3}\gamma_{1j}^{(k)} \\ &\times \frac{\exp\left\{\left[\lambda_j\left(\zeta_1^{(k)}\right)-\lambda_1\left(\zeta_1^{(k)}\right)\right]X\right\}}{\lambda_1\left(\zeta_1^{(k)}\right)-\lambda_1(\zeta)}\Phi_1\left(X,\omega_j\zeta_1^{(k)}\right) \\ &- \sum_{l=1}^{2}\sum_{j=2}^{3}q_{1j}^{(l)} \\ &\times \frac{\exp\{[\lambda_j(\zeta_l')-\lambda_1(\zeta_l')]X\}}{\zeta_l'-\zeta}\Phi_1(X,\omega_j\zeta_l'). \end{aligned} \tag{4.3}$$

As follows from the relationship (4.3) and the formula:

$$\begin{aligned} \phi_{1X}(X,\zeta) = &\frac{i}{\sqrt{3}}[\phi_{1X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) \\ &- \phi_{1X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)], \end{aligned} \tag{4.4}$$

given in [14], for example, the singularities in the form (4.2) appear as poles and in pairs $\zeta_1' = \omega_2\xi_1, \zeta_2' = -\omega_3\xi_1$. From (4.4), considering the limits $\zeta \to \zeta_l'$ and $X \to -\infty$, it also follows immediately that

$$q_{12}^{(1)}\omega_2 = q_{13}^{(2)}. \tag{4.5}$$

We call attention to the fact that, at the special choice of the singularity function $Q_{1j}(\zeta')$ as in (4.2), the second term on the right-hand side of the relation (4.3) is similar in mathematical structure to the third term in this relation (4.3). Indeed, the formal substitution $\xi_2 = i\xi_1, q_{1j}^{(k)} = \gamma_{1j}^{(k)}$ transforms the third term into the second term in (4.3). Then, introducing the notations:

$$\mu_{ji} = \begin{cases} \lambda_j\left(\zeta_1^{(i)}\right) \\ \lambda_j\left(\zeta_{(i-2)}'\right) \end{cases}, \quad p_{1j}^{(i)} = \begin{cases} \gamma_{1j}^{(i)} & \text{at} \quad i=1,2 \\ q_{1j}^{(i-2)} & \text{at} \quad i=3,4 \end{cases}, \tag{4.6}$$

the relationship (4.3) can be rewritten as follows:

$$\begin{aligned} \Phi_1(X,\zeta) = 1 &- \sum_{i=1}^{4}\sum_{j=2}^{3}p_{1j}^{(i)} \\ &\times \frac{\exp\left[\left(\mu_{ji}-\mu_{1i}\right)X\right]}{\mu_{1i}-\zeta}\Phi_1\left(X,\mu_{ji}\right). \end{aligned} \tag{4.7}$$

Since the two terms in (4.3) can be reduced to the same form as in (4.7), we can apply the procedure developed for solving the N-soliton interaction to obtain the solutions connected with the continuum part of the spectral data for the associated eigenvalue problem [14,15]. According to [14] (see Eqs. (5.11–5.15) therein), we can find $\Phi_1(X,\zeta)$ and can connect $\Phi_1(X,\zeta)$ with the solution $W(X)$. As a result, the key relationship

$$U(X) = W_X(X) = 3\frac{\partial^2}{\partial X^2}\ln(\det M(X)) \tag{4.8}$$

can be derived, which is similar to (3.2). Here $M(X)$ is the 4×4 matrix given by

$$M_{il}(X) = \delta_{il} - \sum_{j=2}^{3}p_{1j}^{(i)}\frac{\exp\left[\left(\mu_{ji}-\mu_{1l}\right)X\right]}{\mu_{ji}-\mu_{1l}}. \tag{4.9}$$

Now let us consider the T-evolution of the spectral data. By analyzing the solution of Eq. (2.2) when $X \to -\infty$, we find that $\phi_j(X,T,\zeta) = \exp[-(3\lambda_j(\zeta))^{-1}T]\ \phi_j(X,0,\zeta)$. Hence, the T-evolution of the scattering data is given by the relationships (with $i = 1,2,3,4$):

$$\begin{aligned} &\lambda_j(T) = \lambda_j(0), \\ &p_{1j}^{(i)}(T) = p_{1j}^{(i)}(0)\exp\left\{\left[-(3\mu_{ji})^{-1}+(3\mu_{1i})^{-1}\right]T\right\}. \end{aligned} \tag{4.10}$$

Consequently, the final result for the solution of the VPE, when we consider the spectral data from the bound state spectrum and from the continuous spectrum, as well as taking into account their T-evolution, is as follows:

$$U(X,T) = W_X(X,T) = 3\frac{\partial^2}{\partial X^2}\ln(\det M(X,T)). \tag{4.11}$$

The 4×4 matrix $M(X,T)$ is defined as follows:

$$\begin{aligned} M_{kl} = \delta_{kl} &- \sum_{j=2}^{3}p_{1j}^{(k)} \\ &\times \frac{\exp\left\{(\mu_{jk}-\mu_{1l})X+\left[-(3\mu_{jk})^{-1}+(3\mu_{1k})^{-1}\right]T\right\}}{\mu_{jk}-\mu_{1l}}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} &\mu_{11} = \lambda_1\left(\zeta_1^{(1)}\right) = i\omega_2\xi_1, \quad \mu_{21} = \lambda_2\left(\zeta_1^{(1)}\right) = i\omega_3\xi_1, \\ &p_{12}^{(1)} = \gamma_{12}^{(1)} = \omega_2\beta_1, \quad p_{13}^{(1)} = \gamma_{13}^{(1)} = 0, \\ &\mu_{12} = \lambda_1\left(\zeta_1^{(2)}\right) = -i\omega_3\xi_1, \quad \mu_{32} = \lambda_3\left(\zeta_1^{(2)}\right) = -i\omega_2\xi_1, \\ &p_{12}^{(2)} = \gamma_{12}^{(2)} = 0, \quad p_{13}^{(2)} = \gamma_{13}^{(2)} = \omega_3\beta_1, \\ &\mu_{13} = \lambda_1(\zeta_1') = \omega_2\xi_2, \quad \mu_{23} = \lambda_2(\zeta_1') = \omega_3\xi_2, \\ &p_{12}^{(3)} = q_{12}^{(1)} = \omega_2\beta_2, \quad p_{13}^{(3)} = q_{13}^{(1)} = 0, \\ &\mu_{14} = \lambda_1(\zeta_2') = -\omega_3\xi_2, \quad \mu_{34} = \lambda_3(\zeta_2') = -\omega_2\xi_2, \\ &p_{12}^{(4)} = q_{12}^{(2)} = 0, \quad p_{13}^{(4)} = q_{13}^{(2)} = \omega_3\beta_2. \end{aligned} \tag{4.13}$$

For the solution (4.11) and (4.12) there are two arbitrary constants ξ_i and two arbitrary constants β_i. The constants ξ_i are real, while the constants β_i, in the general case, are complex. The solution obtained through the matrix (4.11)–(4.13) is, in general, a complex function.

As will be clear from the examples in the next section, the solution (4.11)–(4.13) includes a single frequency from the continuum part of the spectral data. For this reason, the solution (4.11)–(4.13), without the soliton, will be referred to as the one-mode solution of the VPE. Evidently this discrete mode emanates from the special choice (4.2) of the singularity functions $Q_{1j}(\zeta')$.

By considering the singularity functions $Q_{1j}(\zeta')$ in (4.2) in the form of the sum of δ-functions, one can generalize without difficulty the solution represented by (4.11)–(4.13) to the interaction of N solitons and K-mode waves. Nevertheless, there is a problem in selecting the real solution from the complex solutions.

4.1. The one-mode solution

In order to obtain the one-mode solution of the VPE (1.1) we need first to calculate the 2×2 matrix $M(X,T)$. For the matrix elements $M_{kl}(X,T)$ we have:

$$M_{11}(X,T)=1-\frac{\mathrm{i}\omega_2\beta_1}{\sqrt{3}\xi_1}\exp\left[-\mathrm{i}\sqrt{3}\xi_1X+\left(\mathrm{i}\sqrt{3}\xi_1\right)^{-1}T\right],$$
$$M_{12}(X,T)=-\frac{\omega_3\beta_1}{2\xi_1}\exp\left[2\omega_3\xi_1X+\left(\mathrm{i}\sqrt{3}\xi_1\right)^{-1}T\right],$$
$$M_{21}(X,T)=\frac{\omega_2\beta_1}{2\xi_1}\exp\left[-2\omega_2\xi_1X+\left(\mathrm{i}\sqrt{3}\xi_1\right)^{-1}T\right],$$
$$M_{22}(X,T)=1-\frac{\mathrm{i}\omega_3\beta_1}{\sqrt{3}\xi_1}\exp\left[-\mathrm{i}\sqrt{3}\xi_1X+\left(\mathrm{i}\sqrt{3}\xi_1\right)^{-1}T\right], \tag{4.14}$$

so that the respective determinant is:

$$\det M(X,T)=\left[1+c_1\exp\left(-\mathrm{i}\sqrt{3}\xi_1X+\left(\mathrm{i}\sqrt{3}\xi_1\right)^{-1}T\right)\right]^2,\quad c_1=\frac{\mathrm{i}\beta_1}{2\sqrt{3}\xi_1}. \tag{4.15}$$

As has already been noted, the singularity functions in the form (4.2) give rise to a single frequency for the continuum part of the spectral data. Hence, the expression (4.15) having been substituted into the concise formula (4.11) must provide us with the one-mode solution.

The condition that W_X is real requires a restriction on the constant β_1 (if the constant ξ_1 is arbitrary, but real). The constant c_1, which in general is the complex-valued one $c_1=|c_1|\exp(\mathrm{i}\chi_1)$, should possess unit modulus $|c_1|=1$, while the arbitrary real constant χ_1 defines an initial shift of solution $X_1=\chi_1/(\sqrt{3}\xi_1)$ so that:

$$\det M(X,T)=\left[1+\exp\left(-\mathrm{i}\sqrt{3}\xi_1(X-X_1)+\frac{T}{\mathrm{i}\sqrt{3}\xi_1}\right)\right]^2. \tag{4.16}$$

The final result for one mode of the continuous spectrum is the solution (4.11) with (4.16), namely:

$$W(X,T)=-3\sqrt{3}\xi_1\tan\left(\frac{\sqrt{3}}{2}\xi_1(X-X_1)+\frac{T}{2\sqrt{3}\xi_1}\right)+\text{const}. \tag{4.17}$$

The corresponding solution for $U=W_X$ (with U governed by (1.4)) was obtained recently by other methods, for example, by the sine–cosine method [17], the (G'/G)-expansion method [13], and the extended tanh-function method [17–19]. However, only the approach developed here and the solution in the form (4.11)–(4.13) enable us to study the interaction of the soliton and a periodic one-mode wave.

4.2. The interaction of a soliton with one-mode wave

The interaction of a soliton with the periodic one-mode wave is described by means of the relation (4.11) with the matrix (4.12) and (4.13). The calculation of this matrix leads to the auxiliary function $F(X,T)=\sqrt{\det M(X,T)}$ as in (3.6), namely

$$F(X,T)=1+c_1q_1+c_2q_2+b_{12}c_1c_2q_1q_2, \tag{4.18}$$

where

$$q_1=\exp\left(\sqrt{3}\xi_1X-\left(\sqrt{3}\xi_1\right)^{-1}T\right),\quad c_1=\frac{\beta_1}{2\sqrt{3}\xi_1},$$
$$q_2=\exp\left(-\mathrm{i}\sqrt{3}\xi_2X+\left(\mathrm{i}\sqrt{3}\xi_2\right)^{-1}T\right),\quad c_2=\frac{\mathrm{i}\beta_2}{2\sqrt{3}\xi_2},$$
$$b_{12}=\left(\frac{\xi_1+\mathrm{i}\xi_2}{\xi_1-\mathrm{i}\xi_2}\right)^2\frac{\xi_1^2-\xi_2^2+\mathrm{i}\xi_1\xi_2}{\xi_1^2-\xi_2^2-\mathrm{i}\xi_1\xi_2},\quad |b_{12}|\equiv 1. \tag{4.19}$$

In Appendix B, the restrictions on the constants c_i for real solutions are found to be:

$$c_1=\pm 1/\sqrt{b_{12}},\quad c_2=\pm 1/\sqrt{b_{12}}. \tag{4.20}$$

The signs here are chosen independently of each other. Consequently, the real solution describing the interaction of one (standard or singular) soliton with the one-mode wave is defined by the relationship (4.11):

$$W(X,T)=6\frac{\partial}{\partial X}\ln(F(X,T))+\text{const}, \tag{4.21}$$

where $F(X,T)$ is (4.18), b_{12} is as in (4.19), while q_i should contain the phaseshifts X_i as in (B.2).

There is an exceptional case for the interaction of one standard soliton with a one-mode wave at $\xi_1=\xi_2$. Then we have $b_{12}=1$, and $F=(1+q_1)(1+q_2)$. Consequently, the solution (4.21) is reduced to the relation

$$W=W_1+W_2=3\sqrt{3}\xi_1\tanh\left(\frac{\sqrt{3}}{2}\xi_1(X-X_1)-\frac{T}{2\sqrt{3}\xi_1}\right)-3\sqrt{3}\xi_1\tan\left(\frac{\sqrt{3}}{2}\xi_1(X-X_2)+\frac{T}{2\sqrt{3}\xi_1}\right)+\text{const}. \tag{4.22}$$

Here W_1 is the one-soliton solution and W_2 is the one-mode solution (4.17). The relationship $W=W_1+W_2$ is easily verified also by the direct substitution in Eq. (1.1). The two waves W_1 and W_2 propagate in opposite directions with the same speed without change of wave profile.

5. Conclusion

In this paper we have applied the inverse scattering method to the Vakhnenko–Parkes equation in order to find the solutions that are associated with both bound state spectrum and continuous spectrum of the spectral problem. We have suggested a special form of the singularity function in order to obtain the periodic solutions. We found sufficient conditions in order that the solutions become real functions. The approach developed here enables us to study the interaction of a soliton and a periodic one-mode wave.

Acknowledgement

The authors are grateful to colleagues from the Bogolyubov Institute for Theoretical Physics of NAS of Ukraine, and especially to Dr. O.O. Vakhnenko for helpful discussions.

Appendix A

Here we consider the conditions on the constants c_i under the interaction of two solitons. Assuming that in the general case the constants $c_i = |c_i|\exp(\mathrm{i}\chi_i)$ are complex-valued, we start with the relationship (3.6) and (3.8), namely:

$$F = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2. \tag{A.1}$$

Let us present the constants c_i in the form:

$$c_i = \alpha_i|c_i|\exp(\mathrm{i}\chi_i) = b_{12}^{-1/2}\exp\left(-\sqrt{3}\xi_iX_i + \mathrm{i}\sigma_i\right),$$
$$\sigma_i = \chi_i + \pi(1-\alpha_i)/2. \tag{A.2}$$

All the new constants χ_i, and $X_i = -\ln(|c_i\sqrt{b_{12}}|)/(\sqrt{3}\xi_i)$ are real. We assume that $-\pi/2 < \chi_i \leqslant \pi/2$, then the values α_i retain the signs of the constants $Re(c_i)$, i.e. $\alpha_i = Re(c_i)/|Re(c_i)|$. It is convenient for analyzing to rewrite (A.1) (the same as (3.6)) in the form:

$$F = 2\exp\left(\theta_1+\theta_2+\frac{\mathrm{i}}{2}(\sigma_1+\sigma_2)\right)G \tag{A.3}$$

with

$$G = \cosh\left(\theta_1+\theta_2+\frac{\mathrm{i}}{2}(\sigma_1+\sigma_2)\right) + b_{12}^{-1/2}\cosh\left(\theta_1-\theta_2+\frac{\mathrm{i}}{2}(\sigma_1-\sigma_2)\right),$$
$$2\theta_i = \sqrt{3}\xi_i(X-X_i) - (\sqrt{3}\xi_i)^{-1}T. \tag{A.4}$$

It is easily seen that only G defines the solution, since $\frac{\partial^2}{\partial X^2}\ln(F) = \frac{\partial^2}{\partial X^2}\ln(G)$, while the conditions that the function G is real are as follows:

$$\chi_i = 0, \quad \sigma_i+\sigma_2 = 2\pi k_1, \quad \sigma_i-\sigma_2 = 2\pi k_2 \tag{A.5}$$

with $k_i = 0,1$. The restrictions (A.5) lead to the results that $\alpha_1 = \pm 1, \alpha_2 = \pm 1$, independently of each other, and $\chi_i = 0$. Then the function F acquires forms:

$$F = 2\exp(\theta_1+\theta_2)G_i, \tag{A.6}$$

where the functions G_i are different for different signs of α_i, namely,

1. for $\alpha_1 = \alpha_2 = 1$:

$$G_1 = \cosh(\theta_1+\theta_2) + b_{12}^{-1/2}\cosh(\theta_1-\theta_2), \tag{A.7}$$

2. for $\alpha_1 = \alpha_2 = -1$:

$$G_2 = \cosh(\theta_1+\theta_2) - b_{12}^{-1/2}\cosh(\theta_1-\theta_2), \tag{A.8}$$

3. for $\alpha_1 = -\alpha_2 = 1$:

$$G_3 = -\sinh(\theta_1+\theta_2) + b_{12}^{-1/2}\sinh(\theta_1-\theta_2), \tag{A.9}$$

4. for $\alpha_1 = -\alpha_2 = -1$:

$$G_4 = -\sinh(\theta_1+\theta_2) - b_{12}^{-1/2}\sinh(\theta_1-\theta_2). \tag{A.10}$$

Hence, the standard soliton solution following from (A.6) and (A.7) and the singular soliton solutions following from (A.6), (A.8),(A.9) and (A.10) are the real functions:

$$U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(G_i). \tag{A.11}$$

Now we rewrite the restrictions in somewhat different form. By retaining the values of phaseshifts X_i in the quantities q_i, we require $c_1 = \pm\sqrt{b_{12}}, c_2 = \pm\sqrt{b_{12}}$, where the signs are independent of each other. Note that for this case there are two arbitrary real constants ξ_i, and two arbitrary real constants $X_i (i = 1,2)$.

The notations (A.7)–(A.10) show that the solution is defined by two combinations of the spectral parameters, namely $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$, but not three values ξ_1, ξ_2, $\xi_1 + \xi_2$ as it may appear from (A.1).

The foregoing proof points to a way for finding the restrictions for any case N. Here it should be noted that the soliton (or singular soliton) solutions are determined by a real function only when c_i is real with either sign of $\alpha_i = c_i/|c_i|$. The conditions on the constants c_i should be:

$$c_i = \pm 1\Bigg/\sqrt{\prod_{\substack{j=1\\ j\neq i}}^{N} b_{ij}}, \quad i = 1,\ldots,N, \tag{A.12}$$

with the retention of the phaseshifts X_i in the quantities q_i. The signs for c_i are independent of each other. The solution will be contained the N real constants ξ_i for determining the values b_{ij} and the N real constants X_i to define the phaseshifts.

Appendix B

In this Appendix we obtain the restrictions on the constants c_i for real solutions taking into account the spectral data from both the bound state spectrum and the continuous spectrum. To find the solution by means of the inverse scattering method, one should know the function (4.18)

$$F(X,T) = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2. \tag{B.1}$$

For convenience we rewrite the variables q_i in the somewhat different form:

$$q_1\exp(2\theta_1), \quad 2\theta_i = \sqrt{3}\xi_i(X-X_i) - (\sqrt{3}\xi_i)^{-1/2}T,$$
$$q_2\exp(\mathrm{i}2\theta_2), \quad 2\theta_2 = -\sqrt{3}\xi_2(X-X_2) - (\sqrt{3}\xi_j)^{-1/2}T. \tag{B.2}$$

The phaseshifts X_i are arbitrary real constants. The value b_{12} in (B.1) are as in (4.19). Note that $b_{12}^* = 1/b_{12}$.

Now we will show that the restrictions:

$$c_1 = \pm\sqrt{b_{12}}, \quad c_2 = \pm\sqrt{b_{12}} \tag{B.3}$$

are sufficient in order to obtain the real solutions. For definiteness, we assume that $\sqrt{b_{12}}$ is a root of an equation $x^2 = b_{12}$ with $-\pi/2 < \arg(\sqrt{b_{12}}) \leqslant \pi/2$. Let us rewrite the relations (B.3) in the form $c_i = \alpha_i\sqrt{b_{12}}$, where $\alpha_i = \pm 1$. It is evident that we can always obtain $\alpha_2 = 1$ by choosing the phaseshift X_2, while we need to consider the two cases $\alpha_1 = \pm 1$. By defining $\sigma = (1 - \alpha_1)/2$, we can rewrite the function F from (B.1) in the form:

$$F(X,T) = 2Ge^{i\pi\sigma}(b_{12})^{-1/4}\exp(\theta_1 + i\pi\sigma/2 + i\theta_2), \tag{B.4}$$

where

$$Ge^{i\pi\sigma} = b_{12}^{1/4}\cos(-i\theta_1 + \pi\sigma/2 + \theta_2) + b_{12}^{-1/4}\cos(-i\theta_1 + \pi\sigma/2 - \theta_2). \tag{B.5}$$

Since $b_{12}^* = 1/b_{12}$ (see (4.19)), it is evident that $G^* = G$, i.e. the variable G constituting the solution is a real-valued function. Hence, with the restrictions (B.3), the solution of the VPE

$$U(X,T) = W_X(X,T) = 6\frac{\partial^2}{\partial X^2}\ln(F) = 6\frac{\partial^2}{\partial X^2}\ln(G) \tag{B.6}$$

represents a real quantity. The signs in (B.3) can be chosen independently of each other. For interaction of one (standard or singular) soliton and a one-mode wave there are two real constants ξ_i and the two real constants X_i.

Note that the restrictions (B.3) are sufficient conditions in order that the solution of the VPE becomes real.

References

[1] Hirota R. The direct method in soliton theory. Cambridge: Cambridge University Press; 2004.

[2] Wazwaz AM. *N*-soliton solutions for the Vakhnenko equation and its generalized forms. Phys Scripta 2010;82:065006(7).

[3] Ablowitz MJ, Segur H. Solitons and the inverse scattering transform. Philadelphia: SIAM; 1981.

[4] Miura RM, editor. Bäcklund transformations, the inverse scattering method, solitons, and their applications. New York: Springer; 1976.

[5] Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. Theory of solitons. The inverse scattering methods. New York, London: Plenum Publishing Corporation; 1984.

[6] Vakhnenko VA. Solitons in a nonlinear model medium. J Phys A Math Gen 1992;25:4181–7.

[7] Parkes EJ. The stability of solutions of Vakhnenkos equation. J Phys A Math Gen 1993;26:6469–75.

[8] Vakhnenko VO. High frequency soliton-like waves in a relaxing medium. J Math Phys 1999;40:2011–20.

[9] Vakhnenko VO, Parkes EJ. The two loop soliton of the Vakhnenko equation. Nonlinearity 1998;11:1457–64.

[10] Morrison AJ, Parkes EJ, Vakhnenko VO. The N loop soliton solution of the Vakhnenko equation. Nonlinearity 1999;12:1427–37.

[11] Gandarias ML, Bruzón MS. Symmmetry relations and exact solutions for the Vakhnenko equation. In: XXI congreso de ecuaciones diferenciales y aplicaciones, XI congreso de matemática aplicada, ciudad real, 21–25 Septiembre, 2009. p. 1–6.

[12] Estévez PG. Reciprocal transformations for a spectral problem in 2 + 1 dimensions. Theor Math Phys 2009;159:763–9.

[13] Abazari R. Application of (G'/G)-expansion method to travelling wave solutions of three nonlinear evolution equation. Comput Fluids 2010;39:1957–63.

[14] Vakhnenko VO, Parkes EJ. The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method. Chaos Soliton Fract 2002;13:1819–26.

[15] Caudrey PJ. The inverse problem for a general $N \times N$ spectral equation. Physica D 1982;6:51–66.

[16] Kaup DJ. On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\ \psi$. Stud Appl Math 1980;62:189–216.

[17] Yusufoglu E, Bekir A. The tanh and the sine-cosine methods for exact solutions of the MBBM and the Vakhnenko equations. Chaos Soliton Fract 2008;38:1126–33.

[18] Parkes EJ. A note on "new travelling wave solutions to the Ostrovsky equation". Appl Math Comput 2010;217:3575–7.

[19] Parkes EJ. Comment on "application of (G′/G)-expansion method to travelling wave solutions of three nonlinear evolution equation" [Comput Fluids 2010;39:1957–1963]. Comput Fluids 2011;42:108–9.