The Inverse Scattering Method for the Equation Describing the High-Frequency Waves in a Relaxing Medium

V.O. Vakhnenko* and E.J. Parkes†

*Institute for Geophysics, Palladin av. 32, Kyiv 03680, Ukraine; E-mail: vakhnenko@bitp.kiev.ua;
†Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK;
E-mail: ejp@maths.strath.ac.uk

Abstract. We suggested the nonlinear evolution equation $(u_t + uu_x)_x + u = 0$ (Vakhnenko equation – VE) for describing the high-frequency perturbations in a relaxing medium. The study of this equation has scientific interest both from the viewpoint of the existence of stable wave formations and from the viewpoint of the general problem of integrability of nonlinear equations. The equation has stable loop-like soliton solutions. The inverse scattering transform (IST) procedure is associated with a third-order eigenvalue problem. This has been achieved by finding a Bäcklund transformation. A procedure for finding the exact *N*-soliton solution to the VE via the IST method is described. Under the interaction of solitons there are features that are not typical for the KdV equation.

EVOLUTION EQUATION FOR HIGH-FREQUENCY WAVES

From the nonequilibrium thermodynamics standpoint, the models of a relaxing medium are more general than the equilibrium models. Thermodynamic equilibrium is disturbed owing to the propagation of fast perturbations. There are processes of the interaction that tend to return the equilibrium. In essence, the change of macroparameters caused by the changes of inner parameters is a relaxation process.

To analyze the wave motion, we use the hydrodynamic equations in Lagrangian coordinaties and dynamic state equation to account for the relaxation effects:

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\rho_0 \partial x} = 0, \qquad \frac{\partial u}{\partial t} + \frac{\partial p}{\rho_0 \partial x} = 0, \qquad d\rho = c_f^{-2} dp + \tau_p^{-1}(\rho - \rho_e)dt \qquad (1.1)$$

The mechanisms of the exchange processes are not defined concretely when deriving the dynamic state. The values c_e, c_f and τ_p can be found experimentally.

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Considering a small nonlinear perturbation $p' \ll p$, from Eqs.(1.1) we obtain the nonlinear evolution equation in one unknown p (the dash in p' is omitted) [1]

$$\tau_p \frac{\partial}{\partial t}\left(\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f \frac{\partial^2 p^2}{\partial t^2}\right) + \left(\frac{\partial^2 p}{\partial x^2} - c_e^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_e \frac{\partial^2 p^2}{\partial t^2}\right) = 0 . \qquad (1.2)$$

For low-frequency perturbations ($\tau_p \omega \ll 1$) Eq.(1.2) is reduced to the Korteweg-de Vries – Burgers (KdVB) equation, while for high-frequency waves ($\tau_p \omega \gg 1$) we have obtained the new equation [1]

$$\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f c_f^2 \frac{\partial^2 p^2}{\partial x^2} + \beta_f \frac{\partial p}{\partial x} + \gamma_e p = 0 . \qquad (1.3)$$

In the general case the last equation has been investigated insufficiently.

Eq.(1.3) without dissipative term can be written down in dimensionless variables $\tilde{x} = \sqrt{\gamma_f/2}(x - c_f t),\ \tilde{t} = \sqrt{\gamma_f/2}c_f t,\ \tilde{u} = \alpha_f c_f^2 p$ (tilde over $\tilde{x}, \tilde{t}, \tilde{u}$ is omitted) [1,2]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \qquad (1.4)$$

Hereafter, as was initiated in [3], this equation is referred to as the Vakhnenko equation (VE). A remarkable feature of Eq.(1.4) is that it has a soliton solution which has loop-like form [2,3]. Whilst loop soliton solutions are rather intriguing, it is the solution to the initial value problem that is of more interest in a physical context.

The physical interpretation of the loop-like soliton solutions was given in [1]. The problem is whether the ambiguity has a physical nature or is related to the incompleteness of the mathematical model, in particular to the lack of dissipation. It is significant that the loop-like solutions are stable to long-wavelength perturbations [3], and that the introduction of a dissipative term does not destroy these loop-like solutions [1]. Consequently, the ambiguity of solution does not relate to the incompleteness of the mathematical model. Thus in the framework of this model approach, the high-frequency perturbation can be described by the multi-valued functions [1].

BÄCKLUND TRANSFORMATION AND LAX PAIR

We have succeeded in finding new coordinates (X,T), in terms of which the solution of Eq.(1.4) is given by single-valued parametric relations [4]

$$x = x_0 + T + W, \quad t = X, \quad W = \int_{-\infty}^{X} U(X',T)dX', \quad u(x,t) = U(X,T). \qquad (2.1)$$

When $X \to -\infty$, the derivatives of W vanish. Eq.(1.4) then has the form [4,5]

$$W_{XXT} + (1+W_T)W_X = 0. \tag{2.2}$$

By taking $W = 6(\ln f)_X$, we observe that the transformed VE (2.2) may be written as the bilinear equation [4,5] through the Hirota binary operator D [6]

$$(D_T D_X^3 + D_X^2) f \cdot f = 0. \tag{2.3}$$

We have obtained a Bäcklund transformation for Eq.(1.6) [5], following [6]

$$(D_X^3 - \lambda(X)) f' \cdot f = 0, \qquad \left(3D_X D_T + 1 + \mu(T) D_X\right) f' \cdot f = 0. \tag{2.4}$$

Separately these two equations appear as part of the Bäcklund transformation for other nonlinear evolution equations. Introducing the function $\psi = f'/f$, we have obtained the following third-order Lax pair for Eq.(2.2)

$$\psi_{XXX} + W_X \psi_X - \lambda\psi = 0, \qquad 3\lambda\psi_T + (1+W_T)\psi_{XX} - W_{XT}\psi_X + \lambda\mu(T)\psi = 0. \tag{2.5}$$

Thus the IST problem is directly related to a spectral equation of third order [5].

INTERACTION OF THE SOLITONS

The third-order eigenvalue problem is similar to the one associated with a higher order KdV equation, a Boussinesq equation, and a model equation for shallow water waves. Kaup [7], Caudrey [8] and Deift et al. [9] studied the inverse problem for certain third-order spectral equations. We adapt the results obtained by these authors to the present problem and describe a procedure for using the IST to find the N-soliton solution to the transformed equation (2.2)

$$U(T,X) = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(T,X)\right), \tag{3.1}$$

where M is the $2N \times 2N$ matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}(0)\frac{\exp\left\{\left[-\left(3\lambda_j(\zeta_1^{(k)})\right)^{-1} + \left(3\lambda_1(\zeta_1^{(k)})\right)^{-1}\right]T + \left(\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})\right)X\right\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})},$$

$$n = 1, 2, \ldots, N, \qquad m = 2n-1,$$

$$\lambda_1(\zeta_1^{(m)}) = \mathrm{i}\omega_2\xi_m, \quad \lambda_2(\zeta_1^{(m)}) = \mathrm{i}\omega_3\xi_m, \quad \gamma_{12}^{(m)}(0) = \omega_2\beta_m, \quad \gamma_{13}^{(m)}(0) = 0,$$

$$\lambda_1(\zeta_1^{(m+1)}) = -\mathrm{i}\omega_3\xi_m, \quad \lambda_3(\zeta_1^{(m+1)}) = -\mathrm{i}\omega_2\xi_m, \quad \gamma_{12}^{(m+1)}(0) = 0, \quad \gamma_{13}^{(m+1)}(0) = \omega_3\beta_m.$$

Here $\lambda_j(\zeta) = \omega_j\zeta$, $\lambda_j^3(\zeta) = \lambda$, and $\omega_j = e^{\mathrm{i}2\pi(j-1)/3}$ are the cube of roots of 1. For the N-soliton solution there are N arbitrary constants ξ_m and N arbitrary constants β_m.

In the interaction of two solitons for the VE (1.4) there are features that are not typical for the KdV equation (see Figs.1 – 2). The larger soliton moving with larger velocity catches up with the smaller soliton moving in the same direction. For convenience in the figures, the interactions of solitons are shown in coordinates moving with the speed of the centre mass. After the nonlinear interaction the solitons separate, their forms are restored, but phaseshifts arise. The larger soliton always has a forward phaseshift, while the smaller soliton can receive any phaseshift. Note that this property is not typical for the KdV equation.

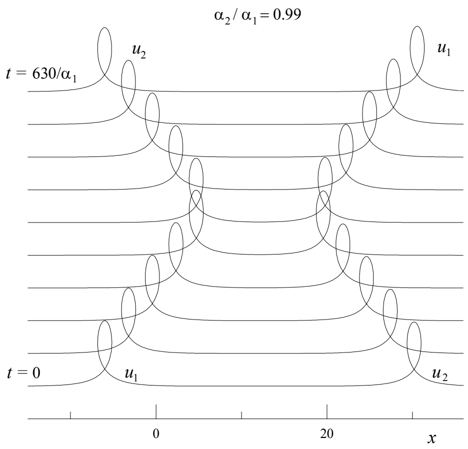

FIGURE 1. Interaction of two solitons in moving coordinates at time $\Delta t = 70 / \alpha_1$.

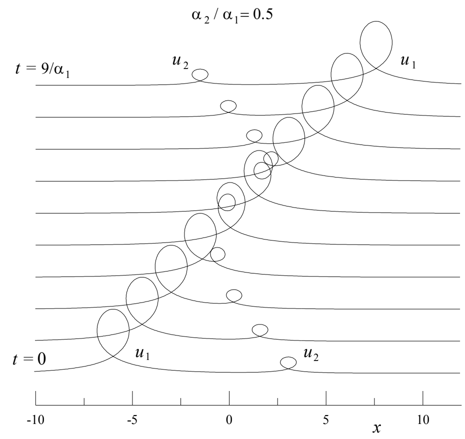

FIGURE 2. Both solitons have phaseshifts in the same direction. Time interval is $\Delta t = 1 / \alpha_1$.

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REFERENCES

1. Vakhnenko, V.O., “High-frequency soliton-like waves in a relaxing medium,” J. Math. Phys. **40**, 2011-2020 (1999).
2. Vakhnenko, V.O., “Solitons in a nonlinear model medium,” J. Phys. A: Math. Gen. **25**, 4181-4187 (1992).
3. Parkes, E.J., “The stability of solutions of Vakhnenko's equation,” J. Phys. A: Math. Gen. **26**, 6469-6475 (1993).
4. Morrison, A.J., *et al.*, “The *N* loop soliton solution of the Vakhnenko equation,” Nonlinearity **12**, 1427-1437 (1999).
5. Vakhnenko, V.O. and Parkes, E.J., “The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method,” Chaos, Solitons and Fractals **13**, 1819-1826 (2002).
6. Hirota, R.. “Direct methods in soliton theory,” in *Solitons,* ed R. K. Bullough and P. J. Caudrey. (Springer, New York, 1980), 157-176.
7. Kaup, D.J., “On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$,” Stud. Appl. Math. **62**, 189-216 (1980).
8. Caudrey P.J., “The inverse problem for a general $N \times N$ spectral equation,” Physica D **6**, 51-66 (1982).
9. Deift P., *et al.*, “Inverse scattering and the Boussinesq equation,” Comm. Pure Appl. Math. **35**, 567-628 (1982).