Nonlinearity **11** (1998) 1457–1464. Printed in the UK PII: S0951-7715(98)89695-3

The two loop soliton solution of the Vakhnenko equation

V O Vakhnenko†§ and E J Parkes‡‖

† Institute of Geophysics, Ukrainian Academy of Sciences, 252054 Kiev, Ukraine

‡ Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK

Received 1 December 1997

Recommended by S Kida

Abstract. An exact two loop soliton solution to the Vakhnenko equation is found. The key step in finding this solution is to transform the independent variables in the equation. This leads to a transformed equation for which it is straightforward to find an exact explicit 2-soliton solution by use of Hirota's method. The exact two loop soliton solution to the Vakhnenko equation is then found in implicit form by means of a transformation back to the original independent variables. The nature of the interaction between the two loop solitons depends on the ratio of their amplitudes.

PACS number: 0340Kf

1. Introduction

In [1] Vakhnenko discussed the nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0 \tag{1.1}$$

which governs the propagation of waves in a relaxing medium [2]. Hereafter (1.1) is referred to as the Vakhnenko equation (VE). Vakhnenko [1] derived two families of periodic travelling-wave solutions to the VE corresponding to propagation in the positive and negative x-direction respectively. In the former case the solutions comprise periodic loops, and there is also a travelling-solitary-wave solution comprising a single loop. Parkes [3] showed that all the aforementioned solutions are stable to long-wavelength perturbations of small amplitude.

Vakhnenko [1] also considered the nonlinear interaction between two solitary waves. However, his formulation of the interaction was in error. The aim of this paper is to revisit this problem; we derive an exact two loop soliton solution to the VE. The key step in finding this solution is to transform the independent variables. This leads to an equation for which it is straightforward to find an exact explicit 2-soliton solution by use of Hirota's method. The exact two loop soliton solution to the VE is then found in implicit form by means of a transformation back to the original independent variables.

The problem discussed in this paper is closely related to work on another equation, namely

$$y_{xt}+\mathrm{sgn}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\left[\frac{y_{xx}}{(1+y_x^2)^{3/2}}\right]_{xx}=0 \tag{1.2}$$

§ E-mail: vakhn@geo.gluk.apc.org

‖ Author to whom correspondence should be addressed. E-mail: e.j.parkes@strath.ac.uk

which describes waves propagating along a stretched rope. Here y is the transverse displacement of the rope and s denotes arc length measured along the solution curve from some reference point on the rope. The inverse scattering method has been used to obtain the one loop soliton solution [4] and two loop soliton solution [5] to (1.2) in implicit form. Ishimori [6] found a transformation of the dependent and independent variables in (1.2) which leads to an mKdV equation in potential form. By use of the known multisoliton solution to the mKdV equation, a multiple loop soliton solution to (1.2) may be constructed [7, 8]. In particular the details of the implicit two and three loop soliton solutions to (1.2) are given explicitly in [8]. The method used in this paper is similar, although here it is only the independent variables that are transformed, and the construction of the loop soliton solutions is more straightforward.

In section 2 the VE is transformed into an equation that has a Hirota form. The previously known one loop soliton solution to the VE is recovered in section 3. The two loop soliton solution to the VE is derived in section 4 and is discussed in section 5; it is found that the nature of the interaction between the loop solitons depends on the ratio of their amplitudes.

2. Transformation of the Vakhnenko equation

We introduce new independent variables X, T defined by

$$x = \theta(X, T) := T + \int_{-\infty}^{X} U(X', T)\, \mathrm{d}X' + x_0 \qquad t = X, \tag{2.1}$$

where $u(x, t) = U(X, T)$, and x_0 is a constant. From (2.1) it follows that

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial T} = \phi\frac{\partial}{\partial x}, \tag{2.2}$$

where

$$\phi(X, T) = 1 + \int_{-\infty}^{X} U_T\, \mathrm{d}X' \tag{2.3}$$

so that

$$\phi_X = U_T. \tag{2.4}$$

From (1.1) and (2.2) we obtain

$$U_{XT} + \phi U = 0. \tag{2.5}$$

By eliminating ϕ between (2.4) and (2.5) we obtain the transformed form of the VE, namely

$$UU_{XXT} - U_X U_{XT} + U^2 U_T = 0. \tag{2.6}$$

In order to find soliton solutions to (2.6) by using Hirota's method [9] we need to express (2.6) in Hirota form. First we introduce W defined by $W_X = U$ and assume that W and its derivatives vanish as $X \to -\infty$. Then $\phi = 1 + W_T$ and (2.5) becomes

$$W_{XXT} + W_X W_T + W_X = 0. \tag{2.7}$$

Equation (2.7) is equivalent to (2.6). By taking

$$W = 6(\ln f)_X, \tag{2.8}$$

we find that

$$W_X = \frac{3D_X^2 f \cdot f}{f^2} \qquad \text{and} \qquad W_{XXT} + W_X W_T = \frac{3D_T D_X^3 f \cdot f}{f^2}, \tag{2.9}$$

where D is the Hirota operator [9], so that (2.7) may be written as the bilinear equation

$$F(D_X, D_T)f \cdot f = 0, \tag{2.10}$$

where

$$F(D_X, D_T) := D_T D_X^3 + D_X^2. \tag{2.11}$$

The solution procedure for the VE is as follows. We solve (2.10) for f by using Hirota's method and hence find the solution $U(X, T)$ to (2.6) by using (2.8). The solution to the VE is then given in parametric form by

$$u(x, t) = U(t, T), \qquad x = \theta(t, T), \tag{2.12}$$

where

$$\theta(X, T) = T + W(X, T) + x_0. \tag{2.13}$$

3. The one loop soliton solution of the Vakhnenko equation

The solution to (2.10) corresponding to one soliton is given by

$$f = 1 + \mathrm{e}^{2\eta}, \qquad \text{where } \eta = kX - \omega T + \alpha, \tag{3.1}$$

and k, ω and α are constants. The dispersion relation is $F(2k, -2\omega) = 0$ from which we find that $\omega = 1/4k$ and then

$$\eta = k(X - cT) + \alpha \qquad \text{with } c = 1/4k^2. \tag{3.2}$$

Substitution of (3.1) into (2.8) gives

$$W(X, T) = 6k(1 + \tanh\eta) \tag{3.3}$$

so that

$$U(X, T) = 6k^2 \operatorname{sech}^2 \eta. \tag{3.4}$$

The one loop soliton solution to the VE is given by (2.12) with (3.3) and (3.4). From (2.13) with $v = 1/c$ we have

$$x - vt = -v(X - cT) + 6k(1 + \tanh[k(X - cT) + \alpha]) + x_0. \tag{3.5}$$

Clearly, from (3.4) and (3.5), $U(X, T)$ and $x - vt$ are related by the parameter $X - cT$ so that $u(x, t)$ is a soliton that travels with speed v in the positive x-direction. That this soliton is a loop may be shown as follows. From (2.2) we have $u_x = \phi^{-1}U_T$ which, together with (2.3), (3.2) and (3.4), yields

$$u_x = -cU_X/(1 - cU). \tag{3.6}$$

Thus, as $X - cT$ goes from $+\infty$ to $-\infty$ in (3.5), so that $x - vt$ goes from $-\infty$ to $+\infty$, U_X changes sign once and remains finite whereas u_x given by (3.6) changes sign three times and goes infinite twice.

If we require symmetry in X–T space, i.e. $U(X, T) = U(-X, -T)$, we take $\alpha = 0$ in (3.2) and then, for symmetry in x–t space, we take $x_0 = -6k$ in (2.13). In this case the one loop soliton solution may be written in terms of the parameter $\zeta := X - cT$ as

$$u = \frac{3v}{2} \operatorname{sech}^2\left(\frac{\sqrt{v}\zeta}{2}\right), \qquad x - vt = 3\sqrt{v}\tanh\left(\frac{\sqrt{v}\zeta}{2}\right) - v\zeta \tag{3.7}$$

with $v (> 0)$ arbitrary. (3.7) is essentially the one loop soliton solution given in [1, 3].

4. The two loop soliton solution of the Vakhnenko equation

The solution to (2.10) corresponding to two solitons is given by

$$f = 1 + e^{2\eta_1} + e^{2\eta_2} + b^2 e^{2(\eta_1+\eta_2)}, \qquad \text{where } \eta_i = k_i X - \omega_i T + \alpha_i, \tag{4.1}$$

$$b^2 = -\frac{F[2(k_1-k_2), -2(\omega_1-\omega_2)]}{F[2(k_1+k_2), -2(\omega_1+\omega_2)]}, \tag{4.2}$$

and k_i, ω_i and α_i are constants. The dispersion relation is $F(2k_i, -2\omega_i) = 0$ from which we find that $\omega_i = 1/4k_i$ and then

$$\eta_i = k_i(X - c_i T) + \alpha_i \qquad \text{with } c_i = 1/4k_i^2. \tag{4.3}$$

Without loss of generality we may take $k_2 > k_1$ and then

$$b = \frac{k_2-k_1}{k_2+k_1}\sqrt{\frac{k_1^2+k_2^2-k_1k_2}{k_1^2+k_2^2+k_1k_2}}, \tag{4.4}$$

so that $0 < b < 1$. Substitution of (4.1) into (2.8) gives $W(X,T)$. Following Hodnett and Moloney [10], we may write $W(X,T)$ in the form

$$W = W_1 + W_2, \qquad \text{where } W_i = 6k_i(1+\tanh g_i) \tag{4.5}$$

and

$$g_1(X,T) = \eta_1 + \frac{1}{2}\ln\left[\frac{1+b^2e^{2\eta_2}}{1+e^{2\eta_2}}\right], \qquad g_2(X,T) = \eta_2 + \frac{1}{2}\ln\left[\frac{1+b^2e^{2\eta_1}}{1+e^{2\eta_1}}\right]. \tag{4.6}$$

It follows that U may be written

$$U = U_1 + U_2, \qquad \text{where } U_i = 6k_i\frac{\partial g_i}{\partial X}\,\text{sech}^2\, g_i. \tag{4.7}$$

The two loop soliton solution to the VE is given by (2.12) with (4.5) and (4.7).

5. Discussion of the two loop soliton solution

We now consider in more detail the two loop soliton solution found in section 4. First it is instructive to consider what happens in X–T space.

As $c_1 > c_2$, we have

$$X - c_2T \to \pm\infty \qquad \text{as } T \to \pm\infty \qquad \text{with } X - c_1T \quad \text{fixed,} \tag{5.1}$$

and

$$X - c_1T \to \mp\infty \qquad \text{as } T \to \pm\infty \qquad \text{with } X - c_2T \quad \text{fixed.} \tag{5.2}$$

From (4.6) and (4.7) with (5.1) it follows that, with $X - c_1T$ fixed,

$$\begin{aligned} U_1 &\sim 6k_1^2\,\text{sech}^2\,\eta_1 \qquad \text{as } T \to -\infty, \\ U_1 &\sim 6k_1^2\,\text{sech}^2(\eta_1 + \ln b) \qquad \text{as } T \to +\infty. \end{aligned} \tag{5.3}$$

Similarly, from (4.6) and (4.7) with (5.2), with $X - c_2T$ fixed,

$$\begin{aligned} U_2 &\sim 6k_2^2\,\text{sech}^2(\eta_2 + \ln b) \qquad \text{as } T \to -\infty, \\ U_2 &\sim 6k_2^2\,\text{sech}^2\,\eta_2 \qquad \text{as } T \to +\infty. \end{aligned} \tag{5.4}$$

Hence it is apparent that, in the limits $T \to \pm\infty$, U_1 and U_2 may be identified as individual solitons moving with speeds c_1 and c_2 respectively in the positive X-direction. In contrast to the familiar interaction of two KdV 'sech-squared' solitons [11], here it is the smaller soliton that overtakes the larger one. The shifts, Δ_i, of the two solitons U_1 and U_2 in the positive X-direction due to the interaction are

$$\Delta_1 = -(\ln b)/k_1 \qquad \text{and} \qquad \Delta_2 = (\ln b)/k_2 \tag{5.5}$$

respectively. As $\ln b < 0$, the smaller soliton is shifted forwards and the larger soliton is shifted backwards. Since the 'mass' of each soliton is given by $\int_{-\infty}^{\infty} U_i \, \mathrm{d}X = 12k_i$, where we have used (4.7), and the shifts satisfy $k_1\Delta_1 + k_2\Delta_2 = 0$, 'momentum' is conserved.

Let $r := k_1/k_2$ and recall that here we are assuming that $0 < r < 1$. (From (5.3) and (5.4), r^2 is the ratio of the amplitudes of the individual smaller and larger solitons.) Note that $U_{XX}(X_{\mathrm{int}}, T_{\mathrm{int}}) = 0$ for $r = R = 0.538\,62$, where $(X_{\mathrm{int}}, T_{\mathrm{int}})$ is the centre of the interaction. (If the condition (5.9) is satisfied then $X_{\mathrm{int}} = 0$ and $T_{\mathrm{int}} = 0$.) For $R < r < 1$, we have $U_{XX}(X_{\mathrm{int}}, T_{\mathrm{int}}) > 0$ and the 2-soliton solution in X–T space always has two peaks; during interaction the two humps exchange amplitudes. For $0 < r < R$, we have $U_{XX}(X_{\mathrm{int}}, T_{\mathrm{int}}) < 0$ and the two humps of the individual solitons coalesce into a single hump for part of the interaction; the smaller hump appears to pass through the larger one.

Now let us consider what happens in x–t space. From (2.13) with $v_i = 1/c_i$ we have

$$x - v_i t = -v_i(X - c_i T) + W(X, T) + x_0. \tag{5.6}$$

Note that in (5.3) taking the limits $T \to \pm\infty$ with $X - c_1 T$ fixed is equivalent to taking the limits $X \to \pm\infty$ with $X - c_1 T$ fixed; also note that $X = t$ from (2.1). Accordingly from (5.3) and (5.6) with $i = 1$ we see that in the limits $t \to \pm\infty$ with $X - c_1 T$ fixed, $U_1(X, T)$ and $x - v_1 t$ are related by the parameter $X - c_1 T$. Similarly, from (5.4) and (5.6) with $i = 2$, in the limits $t \to \pm\infty$ with $X - c_2 T$ fixed, $U_2(X, T)$ and $x - v_2 t$ are related by the parameter $X - c_2 T$. It follows that in the limits $t \to \pm\infty$, u_1 and u_2 may be identified as individual loop solitons moving with speeds v_1 and v_2 respectively in the positive x-direction, where $u_i(x, t) = U_i(X, T)$. As $v_2 > v_1$, the larger loop soliton overtakes the smaller loop soliton. This is in contrast to the two loop soliton solution to (1.2) in which the smaller loop soliton overtakes the larger one [5].

The shifts, δ_i, of the two loop solitons u_1 and u_2 in the positive x-direction due to the interaction may be computed from (5.6) as follows. From (5.3), as $T \to -\infty$, $U_1 = U_{1\max} = 6k_1^2$ where $X - c_1 T = -\alpha_1/k_1$; then $W_1 = 6k_1$ and, by use of (5.1), $W_2 = 0$. Similarly, as $T \to \infty$, $U_1 = U_{1\max} = 6k_1^2$ where $X - c_1 T = -(\alpha_1 + \ln b)/k_1$; then $W_1 = 6k_1$ and $W_2 = 12k_2$. Use of these results in (5.6) with $i = 1$ gives

$$\delta_1 = 4k_1 \ln b + 12k_2. \tag{5.7}$$

By use of (5.2), (5.4) and (5.6) with $i = 2$, a similar calculation yields

$$\delta_2 = -4k_2 \ln b - 12k_1. \tag{5.8}$$

Plots of δ_1/k_2 and δ_2/k_2 as functions of r with $0 < r < 1$ are shown in figure 1. It may be seen that $\delta_2 > 0$ so that the larger loop soliton is always shifted forwards by the interaction. However, for δ_1 we find that:

(a) for $r = r_c$, where $r_c = 0.888\,67$ is the root of $\ln b + 3/r = 0$, $\delta_1 = 0$ so the smaller loop soliton is not shifted by the interaction;

(b) for $0 < r < r_c$, $\delta_1 > 0$ so the smaller loop soliton is shifted forwards;

(c) for $r_c < r < 1$, $\delta_1 < 0$ so the smaller loop soliton is shifted backwards.

At first sight it might seem that the behaviour in (a) and (b) contradicts conservation of 'momentum'. That this is not so is justified as follows. By integrating (1.1) with respect to x we find that $\int_{-\infty}^{\infty} u \, \mathrm{d}x = 0$; also, by multiplying (1.1) by x and integrating with respect to x we obtain $\int_{-\infty}^{\infty} xu \, \mathrm{d}x = 0$. Thus, in x–t space, the 'mass' of each soliton is zero, and

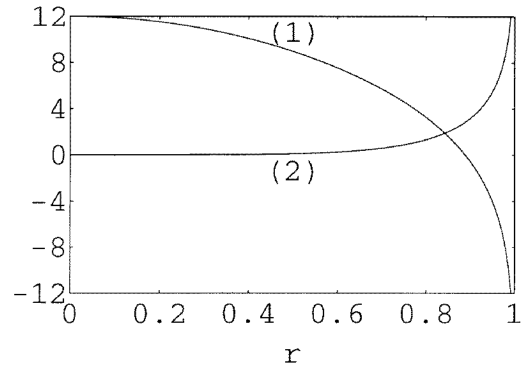

Figure 1. The quantities δ_1/k_2 (curve 1) and δ_2/k_2 (curve 2), given by (5.7) and (5.8) respectively, as functions of r.

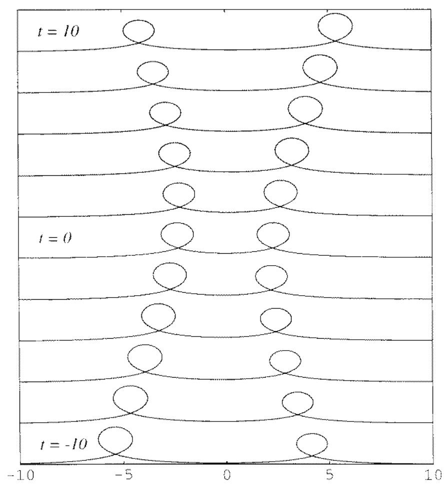

Figure 2. The interaction process for two loop solitons with $k_1 = 0.9$ and $k_2 = 1$ so that $r = 0.9$ and $\delta_1 < 0$.

'momentum' is conserved whatever δ_1 and δ_2 may be. In particular δ_1 and δ_2 may have the same sign as in (b), or one of them may be zero as in (a).

For the interaction to be centred at $X = 0$, $T = 0$ we require

$$\alpha_1 = \alpha_2 = -\tfrac{1}{2}\ln b \tag{5.9}$$

and then, for the interaction to be centred at $x = 0$, $t = 0$ we require

$$x_0 = -6k_1 - 6k_2. \tag{5.10}$$

We have used (5.9) and (5.10) in the computation of figures 2–4.

We have already seen that the shifts given by (5.7) and (5.8) depend upon the ratio r. The behaviour of the solution during the interaction process also depends on r. There are three characteristic cases:

(1) for $r_b < r < 1$, where $r_b = 0.759\,68$, the two loops exchange their amplitudes during the interaction but never overlap;

(2) for $r_a < r < r_b$, where $r_a = 0.556\,76$, the two loops exchange their amplitudes during the interaction and, for part of the interaction, the loops overlap;

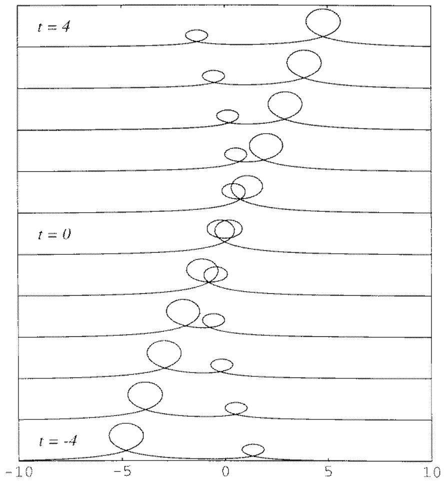

Figure 3. The interaction process for two loop solitons with $k_1 = 0.65$ and $k_2 = 1$ so that $r = 0.65$ and $\delta_1 > 0$.

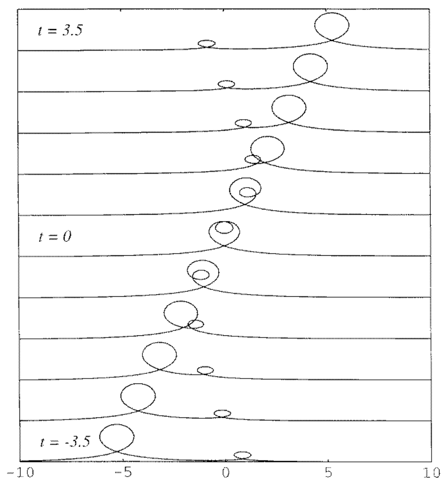

Figure 4. The interaction process for two loop solitons with $k_1 = 0.5$ and $k_2 = 1$ so that $r = 0.5$ and $\delta_1 > 0$.

(3) for $0 < r < r_a$, the larger loop catches up the smaller loop which then travels clockwise around the larger loop before being ejected behind the larger loop.

Cases (1)–(3) are illustrated in figures 2–4 respectively; in each of these figures u is plotted against $x - (v_1 + v_2)t/2$ at several equally spaced values of t.

6. Conclusion

We have found the two loop soliton solution to the VE by using a blend of transformations and Hirota's method. The procedure can also be used to find N loop soliton solutions with $N > 2$. This, together with a detailed investigation of the case $N = 3$, will be reported elsewhere. It should be possible to find loop soliton solutions by use of the inverse scattering transform method [11]. This is currently under investigation.

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