

## Solutions Associated with Discrete and Continuous Spectrums in the Inverse Scattering Method for the Vakhnenko-Parkes Equation

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In this paper the inverse scattering method is applied to the Vakhnenko-Parkes equation. We describe a procedure for using the inverse scattering transform to find the solutions that are associated with both the bound state spectrum and continuous spectrum of the spectral problem. The suggested special form of the singularity function gives rise to the multi-mode periodic solutions. Sufficient conditions are obtained in order that the solutions become real functions. The interaction of the solitons and multi-mode periodic waves is studied. The procedure is illustrated by considering a number of examples.

Subject Index: 010, 011

### §1. Introduction

It is of significance to look for exact solutions of nonlinear evolution equations in many applications of physics and technology. Various effective approaches have been developed to construct exact wave solutions of completely integrable equations. One of the fundamental direct methods is undoubtedly the Hirota bilinear method,<sup>1),2)</sup> which possesses significant features that make it practical for the determination of multiple-soliton solutions. However, the direct methods can be applied only for finding solitary-wave solutions or traveling-wave solutions. In this sense, the inverse scattering method is the most appropriate way of tackling the initial value problem although its employment is a fairly difficult procedure.<sup>3)-5)</sup>

This paper deals with a nonlinear evolution equation

$$W_{XXT} + (1 + W_T)W_X = 0. \quad (1.1)$$

This equation arises from the Vakhnenko equation<sup>6)-8)</sup>

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) u + u = 0 \quad (1.2)$$

through the transformation<sup>9),10)</sup>

$$u(x, t) := U(X, T) = W_X(X, T), \quad x := x_0 + T + W(X, T), \quad t := X. \quad (1.3)$$

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The corresponding governing equation for  $U$ , namely

$$UU_{XXT} - U_XU_{XT} + U^2U_T = 0 \quad (1.4)$$

is given in the paper.<sup>9)</sup>

Equations (1.1), (1.2), and (1.4) arose as a result of describing high-frequency perturbations in a relaxing medium.<sup>8)</sup> Following the papers,<sup>11),12)</sup> hereafter the equation (1.1) is referred to as the Vakhnenko-Parkes equation (VPE).

Recently the Hirota method<sup>2),9)</sup> as well as the inverse scattering method<sup>13)</sup> have been applied to obtain the exact  $N$ -soliton solutions of the VPE. In this paper we use the inverse scattering transform method to study additionally the periodic solutions of the VPE (1.1) associated with the continuous part of the spectral data as well as to investigate the interaction of solitons with these periodic waves.

In §2 we formulate the spectral problem for the VPE by adapting the results given by Caudrey<sup>14)</sup> and by Kaup.<sup>15)</sup> In §3 we find the solutions corresponding to both the bound state and continuous part of the spectral data. In §4 we find the corresponding real  $M$ -soliton and periodic  $N$ -mode solutions and study their interactions. Our results are summarized in §5.

## §2. The inverse spectral problem for the VPE

In order to use the inverse scattering method, one first has to formulate the associated eigenvalue problem. In the paper,<sup>13)</sup> it is shown that the pair of equations

$$\psi_{XXX} + W_X\psi_X - \lambda\psi = 0, \quad (2.1)$$

$$3\psi_{XT} + (W_T + 1)\psi = 0 \quad (2.2)$$

is associated with the VPE (1.1) considered here through the quantity  $W$ . Indeed, the compatibility of Eqs. (2.1) and (2.2) yields the condition<sup>13)</sup>  $(W_{XXT} + (1 + W_T)W_X)_X = 0$  or  $W_{XXT} + (1 + W_T)W_X + h(T) = 0$ , where  $h(T)$  is an arbitrary function of  $T$ . Now, according to (2.6) and (3.10), the inverse scattering method restricts the solutions to those that vanish as  $|X| \rightarrow \infty$ , so  $h(T)$  is to be identically zero. Thus, the pair of equations (2.1) and (2.2) can be considered as the Lax pair for the VPE (1.1).

Note that the inverse scattering transform problem is related to a spectral equation of third order (2.1). The inverse problem for third-order spectral equations has been considered by Caudrey<sup>14)</sup> and Kaup.<sup>15)</sup> We adapt the results obtained by these authors to the present spectral problem and describe a procedure for using the inverse scattering transform method to find the solutions of the VPE.

We follow the general theory of the inverse scattering problem for  $N$  spectral equations which has been developed by Caudrey.<sup>14)</sup> According to this paper<sup>14)</sup> the spectral equation (2.1) can be rewritten in the form<sup>13)</sup>

$$\frac{\partial}{\partial X}\psi = [\mathbf{A}(\zeta) + \mathbf{B}(X, \zeta)] \cdot \psi \quad (2.3)$$

with

$$\psi = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -W_X & 0 \end{pmatrix}. \quad (2.4)$$

The matrix  $\mathbf{A}$  has the eigenvalues  $\lambda_j(\zeta)$  and left- and right-eigenvectors  $\tilde{\mathbf{v}}_j(\zeta)$  and  $\mathbf{v}_j(\zeta)$ , respectively, ( $j = 1, 2, 3$ ). These quantities are defined through a spectral parameter  $\lambda$  as

$$\lambda_j(\zeta) = \omega_j \zeta, \quad \lambda_j^3(\zeta) = \lambda, \quad \mathbf{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j(\zeta) \\ \lambda_j^2(\zeta) \end{pmatrix}, \quad \tilde{\mathbf{v}}_j(\zeta) = (\lambda_j^2(\zeta) \quad \lambda_j(\zeta) \quad 1), \quad (2.5)$$

where  $\omega_j = e^{2\pi i(j-1)/3}$  are the cube roots of 1.

The solution of the linear equation (2.1), or equivalently Eq. (2.3), has been obtained by Caudrey<sup>14)</sup> in terms of Jost functions  $\phi_j(X, \zeta)$  which have the asymptotic behaviour

$$\Phi_j(X, \zeta) := \exp\{-\lambda_j(\zeta)X\} \phi_j(X, \zeta) \rightarrow \mathbf{v}_j(\zeta) \quad \text{as } X \rightarrow -\infty. \quad (2.6)$$

Here  $T$  is regarded as a parameter; the  $T$ -evolution of the scattering data will be taken into account later. The solution of the direct problem (2.3) is given by the equation system (4.5) in the paper.<sup>14)</sup> Since there is a set of symmetry properties  $\phi_1(X, \zeta/\omega_1) = \phi_2(X, \zeta/\omega_2) = \phi_3(X, \zeta/\omega_3)$  (see e.g. (6.14) and (6.15) in the paper<sup>14)</sup>) for Jost functions  $\phi_j(X, \zeta)$ , we need only consider the element  $\phi_1(X, \zeta)$  (as well as  $\Phi_1(X, \zeta)$ ). In the general case it is necessary to take into account both the bound state spectrum and the continuous spectrum. According to the relation (6.20) in the paper,<sup>14)</sup> the solution of (2.3) is as follows:

$$\begin{aligned} \Phi_1(X, \zeta) = & 1 - \sum_{k=1}^K \sum_{j=2}^3 \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Phi_1(X, \omega_j \zeta_1^{(k)}) \\ & + \frac{1}{2\pi i} \int \sum_{j=2}^3 Q_{1j}(\zeta') \frac{\exp\{[\lambda_j(\zeta') - \lambda_1(\zeta')]X\}}{\zeta' - \zeta} \Phi_1^\pm(X, \omega_j \zeta') d\zeta'. \end{aligned} \quad (2.7)$$

Equation (2.7) contains the spectral data, namely,  $K$  poles with the quantities  $\gamma_{1j}^{(k)}$  for the bound state spectrum as well as the functions  $Q_{1j}(\zeta')$  given along all the boundaries of regular regions for the continuous spectrum. The boundaries between regions, where the Jost function  $\phi_1(X, \zeta)$  is regular, appear at  $\text{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$  over all  $j \neq 1$ .<sup>14)</sup> The singularities on boundaries of these regions within the complex  $\zeta$ -plane are taken into account by the third term in the relation (2.7). The integral in (2.7) is along all the boundaries (see the dashed lines in Fig. 1). The direction of integration is taken so that the side chosen to be  $\text{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) < 0$  is shown by the arrows in Fig. 1. It is necessary to note that we should carry out the integration along the lines  $\omega_2(\xi + i\varepsilon)$  and  $-\omega_3(\xi + i\varepsilon)$  with  $\varepsilon > 0$ . In this case, as we will show in §4.1.1, the condition (2.6) is satisfied. Passing to the limit  $\varepsilon \rightarrow 0$  we can obtain the

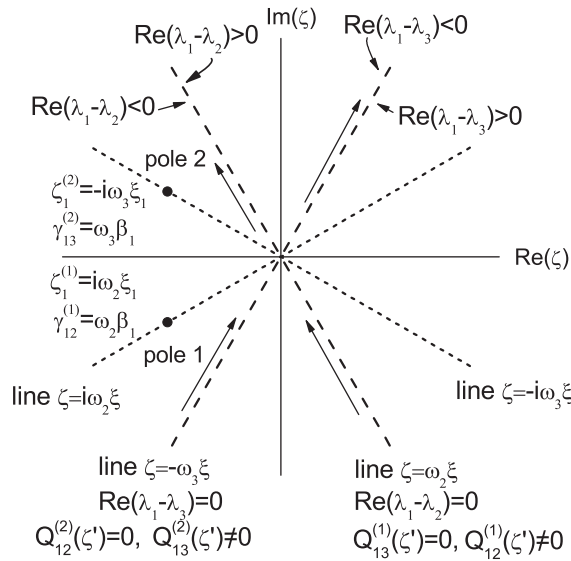


Fig. 1. The regular regions for Jost functions  $\phi_1(X, \zeta)$  in the complex  $\zeta$ -plane. The dashed lines determine the boundaries between regular regions. These lines are lines where the singularity functions  $Q_{1,j}(\zeta')$  are given. The dotted lines are the lines where the poles appear.

solution which does not satisfy the condition (2.6) (see §4.1). However, for any finite  $\varepsilon > 0$  the restricted region on  $X$  can be determined where the solution associated with a finite  $\varepsilon > 0$  (for which the condition (2.6) is valid) and the solution associated with  $\varepsilon = 0$  are sufficiently close to each other (see §4.1.1 for details). In this sense, taking the integration at  $\varepsilon = 0$ , we remain within the inverse scattering theory<sup>14)</sup> whereas the condition (2.6) can be omitted. The solution obtained at  $\varepsilon = 0$  can be extended to sufficiently large finite  $X$ . Thus, we will interpret the solution obtained at  $\varepsilon = 0$  as the solution of Eq. (1.1) which is valid on arbitrary but finite  $X$ .

Provided  $Q_{1j}(\zeta) \equiv 0$  in (2.7), the consideration of the bound state spectrum only gives rises to the purely soliton solutions. The procedure for finding the exact  $N$ -soliton solution of the VPE via the inverse scattering method is described in the paper.<sup>13)</sup> In the next section we study the solutions of the VPE taking into account additionally the continuous part of the spectral data.

### §3. The soliton and periodic solutions

Now additionally to the bound state spectrum we consider the continuous spectrum of the associated eigenvalue problem, i.e. assume that at least some of the functions  $Q_{1j}(\zeta')$  are nonzero. At each fixed  $j \neq 1$  the functions  $Q_{1j}(\zeta')$  characterize the singularity of  $\Phi_1(X, \zeta)$ . This singularity can appear only on boundaries between

the regular regions on the  $\zeta$ -plane. The condition  $\text{Re}(\lambda_1(\zeta') - \lambda_j(\zeta')) = 0$  determines these boundaries.<sup>14)</sup> According to the paper,<sup>14)</sup> we find that for  $\Phi_1(X, \zeta)$  the complex  $\zeta$ -plane is divided into four regions by two lines

$$\begin{aligned} \text{(i)} \quad & \zeta' = \omega_2 \xi, \quad \text{with } Q_{12}^{(1)}(\zeta') \neq 0, \quad Q_{13}^{(1)}(\zeta') \equiv 0, \\ \text{(ii)} \quad & \zeta' = -\omega_3 \xi, \quad \text{with } Q_{12}^{(2)}(\zeta') \equiv 0, \quad Q_{13}^{(2)}(\zeta') \neq 0, \end{aligned} \tag{3.1}$$

where  $\xi$  is real (see Fig. 1). Analysis shows that the direction of the integration in (2.7) is to be so that  $\xi$  sweeps from  $-\infty$  to  $+\infty$ .

Let us consider the singularity functions  $Q_{1j}(\zeta')$  on the boundaries, on which the Jost function  $\phi_1(X, \zeta)$  is singular, in the form ( $n = 1, 2, \dots, N$ )

$$\left. \begin{aligned} Q_{12}^{(1)}(\zeta') &= -2\pi i \sum_{n=1}^N q_{12}^{(2n-1)} \delta(\zeta' - \zeta'_{2n-1}) \\ Q_{13}^{(1)}(\zeta') &= -2\pi i \sum_{n=1}^N q_{13}^{(2n-1)} \delta(\zeta' - \zeta'_{2n-1}) \equiv 0 \end{aligned} \right\} \text{ on the line } \zeta' = \omega_2 \xi,$$

$$\left. \begin{aligned} Q_{12}^{(2)}(\zeta') &= -2\pi i \sum_{n=1}^N q_{12}^{(2n)} \delta(\zeta' - \zeta'_{2n}) \equiv 0 \\ Q_{13}^{(2)}(\zeta') &= -2\pi i \sum_{n=1}^N q_{13}^{(2n)} \delta(\zeta' - \zeta'_{2n}) \end{aligned} \right\} \text{ on the line } \zeta' = -\omega_3 \xi. \tag{3.2}$$

For the singularity functions (3.2) and for  $M$  pairs of poles, the relationship (2.7) is reduced to the form

$$\begin{aligned} \Phi_1(X, \zeta) &= 1 - \sum_{k=1}^{2M} \sum_{j=2}^3 \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Phi_1(X, \omega_j \zeta_1^{(k)}) \\ &\quad - \sum_{l=1}^{2N} \sum_{j=2}^3 q_{1j}^{(l)} \frac{\exp\{[\lambda_j(\zeta'_l) - \lambda_1(\zeta'_l)]X\}}{\zeta'_l - \zeta} \Phi_1(X, \omega_j \zeta'_l). \end{aligned} \tag{3.3}$$

In the paper,<sup>13)</sup> it is proved that the poles appear in pairs only  $\zeta_1^{(2m-1)} = i\omega_2 \xi_1$  and  $\zeta_1^{(2m)} = -i\omega_3 \xi_1$ , under the conditions  $\gamma_{12}^{(2m-1)} = \omega_2 \beta_m$ ,  $\gamma_{13}^{(2m-1)} = 0$ ,  $\gamma_{12}^{(2m)} = 0$ , and  $\gamma_{13}^{(2m)} = \omega_3 \beta_m$ , ( $m = 1, 2, \dots, M$ ). If we consider both the bound state spectrum and the continuous spectrum, the constants  $\beta_m$  are complex values in the general case. The restrictions on the constants  $\beta_m$  for real solutions  $U$  follow from a separate problem, which will be analyzed in the next section.

As follows from the relationship (3.3) and the formula

$$\phi_{1X}(X, \zeta) = \frac{i}{\sqrt{3}} [\phi_{1X}(X, -\omega_2 \zeta) \phi_1(X, -\omega_3 \zeta) - \phi_{1X}(X, -\omega_3 \zeta) \phi_1(X, -\omega_2 \zeta)] \tag{3.4}$$

given in the paper,<sup>13)</sup> for example, the singularities in the form (3.2) appear in pairs  $\zeta'_{2n-1} = \omega_2 \xi_n$  and  $\zeta'_{2n} = -\omega_3 \xi_n$ . From (3.4), on considering the limits  $\zeta \rightarrow \zeta'_l$  and  $X \rightarrow -\infty$ , it immediately follows that

$$q_{12}^{(2n-1)} \omega_2 = q_{13}^{(2n)} \quad \text{for } n = 1, 2, \dots, N. \tag{3.5}$$

Insofar as we have  $2M$  poles and  $2N$  coefficients  $q_{12}^{(2n-1)}$  and  $q_{13}^{(2n)}$  in the adopted specifications (3.2) of the singularity functions  $Q_{1j}(\zeta')$ , it is convenient to introduce the notation

$$\mu_{ji} = \begin{cases} \lambda_j(\zeta_1^{(i)}), \\ \lambda_j(\zeta'_{(i-K)}), \end{cases} \quad p_{1j}^{(i)} = \begin{cases} \gamma_{1j}^{(i)} & \text{at } i = 1, \dots, K, \\ q_{1j}^{(i-K)} & \text{at } i = K + 1, \dots, K + L, \end{cases} \quad (3.6)$$

where  $K = 2M$  and  $L = 2N$ . Then the relationship (3.3) is rewritten as follows:

$$\Phi_1(X, \zeta) = 1 - \sum_{i=1}^{K+L} \sum_{j=2}^3 p_{1j}^{(i)} \frac{\exp[(\mu_{ji} - \mu_{1i})X]}{\mu_{1i} - \zeta} \Phi_1(X, \mu_{ji}). \quad (3.7)$$

By defining

$$\Psi_i(X) = \sum_{j=2}^3 p_{1j}^{(i)} \exp(\mu_{ji}X) \Phi_1(X, \mu_{ji}), \quad (3.8)$$

we may rewrite the relationship (3.7) as

$$\Phi_1(X, \zeta) = 1 - \sum_{i=1}^{K+L} \frac{\exp(-\mu_{1i}X)}{\mu_{1i} - \zeta} \Psi_i(X). \quad (3.9)$$

On the other hand, by expanding  $\Phi_1(X, \zeta)$  as an asymptotic series in  $\lambda_1^{-1}(\zeta)$ , one can obtain (see Eq. (5.11) in the paper<sup>13</sup>)

$$\Phi_1(X, \zeta) = 1 - \frac{1}{3\lambda_1(\zeta)} [W(X) - W(-\infty)] + O(\lambda_1^{-2}(\zeta)). \quad (3.10)$$

From (3.8) and (3.9), the following key relationship may be found (see also Eq. (6.38) in the paper<sup>14</sup>):

$$W(X) - W(-\infty) = -3 \sum_{k=1}^{K+L} \exp(-\mu_{1k}X) \Psi_k(X) = 3 \frac{\partial}{\partial X} \ln(\det M). \quad (3.11)$$

Here the matrix  $M(X)$  is defined as follows:

$$M_{il}(X) = \delta_{il} - \sum_{j=2}^3 p_{1j}^{(i)} \frac{\exp[(\mu_{ji} - \mu_{1l})X]}{\mu_{ji} - \mu_{1l}}. \quad (3.12)$$

Now let us consider the  $T$ -evolution of the spectral data. By analyzing the solution of Eq. (2.2) when  $X \rightarrow -\infty$ , we find that  $\phi_j(X, T, \zeta) = \exp[-(3\lambda_j(\zeta))^{-1}T] \times \phi_j(X, 0, \zeta)$ . Hence, the  $T$ -evolution of the scattering data is given by the relationships (with  $i = 1, 2, \dots, K + L$ )

$$\lambda_j(T) = \lambda_j(0), \quad p_{1j}^{(i)}(T) = p_{1j}^{(i)}(0) \exp\{[-(3\mu_{ji})^{-1} + (3\mu_{1i})^{-1}]T\}. \quad (3.13)$$

Consequently, the final result for the solution of the VPE, when we consider the spectral data from both the bound state spectrum and the continuous spectrum, as well as taking into account their  $T$ -evolution, is as follows:

$$U(X, T) = W_X(X, T) = 3 \frac{\partial^2}{\partial X^2} \ln (\det M(X, T)). \tag{3.14}$$

Here  $M(X, T)$  is the  $(K + L) \times (K + L)$  matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^3 p_{1j}^{(k)} \frac{\exp\{(\mu_{jk} - \mu_{1l})X + [-(3\mu_{jk})^{-1} + (3\mu_{1k})^{-1}]T\}}{\mu_{jk} - \mu_{1l}}, \tag{3.15}$$

where, for  $i \leq M$ ,

$$\begin{aligned} \mu_{1(2i-1)} &= \lambda_1(\zeta_1^{(2i-1)}) = i\omega_2\xi_i, & \mu_{2(2i-1)} &= \lambda_2(\zeta_1^{(2i-1)}) = i\omega_3\xi_i, \\ p_{12}^{(2i-1)} &= \gamma_{12}^{(2i-1)} = \omega_2\beta_i, & p_{13}^{(2i-1)} &= \gamma_{13}^{(2i-1)} = 0, \\ \mu_{1(2i)} &= \lambda_1(\zeta_1^{(2i)}) = -i\omega_2\xi_i, & \mu_{3(2i)} &= \lambda_3(\zeta_1^{(2i)}) = -i\omega_2\xi_i, \\ p_{12}^{(2i)} &= \gamma_{12}^{(2i)} = 0, & p_{13}^{(2i)} &= \gamma_{13}^{(2i)} = \omega_3\beta_i, \end{aligned} \tag{3.16}$$

and for  $M < i \leq M + N$ ,

$$\begin{aligned} \mu_{1(2i-1)} &= \lambda_1(\zeta'_{2(i-M)-1}) = \omega_2\xi_i, & \mu_{2(2i-1)} &= \lambda_2(\zeta'_{2(i-M)-1}) = \omega_3\xi_i, \\ p_{12}^{(2i-1)} &= q_{12}^{(2(i-M)-1)} = \omega_2\beta_i, & p_{13}^{(2i-1)} &= q_{13}^{(2(i-M)-1)} = 0, \\ \mu_{1(2i)} &= \lambda_1(\zeta'_{2(i-M)}) = -\omega_3\xi_i, & \mu_{3(2i)} &= \lambda_3(\zeta'_{2(i-M)}) = -\omega_2\xi_i, \\ p_{12}^{(2i)} &= q_{12}^{(2(i-M))} = 0, & p_{13}^{(2i)} &= q_{13}^{(2(i-M))} = \omega_3\beta_i. \end{aligned} \tag{3.17}$$

For the solution (3.14), (3.15) there are  $(M + N)$  arbitrary constants  $\xi_i$  and  $(M + N)$  arbitrary constants  $\beta_i$ . The constants  $\xi_i$  are real, while the constants  $\beta_i$ , in the general case, are complex.

As will be clear from the examples in the next section, the solution (3.14), (3.15) includes  $N$  discrete frequencies from the continuous part of the spectral data. For this reason, the solution (3.14), (3.15), without solitons (i.e. with  $M = 0$ ), will be referred to as  $N$ -mode solution of the VPE. Evidently these discrete modes emanate from the special choice (3.2) of the singularity functions  $Q_{1j}(\zeta')$ .

The solution obtained through the matrix (3.15) is in general a complex function. Consequently, there is a problem in selecting the real solutions from the complex solutions. It turns out that we can obtain the real solutions by means of restriction of arbitrariness in the choice of the constants  $\beta_i$ . We have succeeded in finding these restrictions.

#### §4. Real solutions for the VPE

The goal of this research is to find real functions for  $U = W_X$ . We analyze a number of examples, as well as the general case, for the interaction of the solitons and multi-mode waves. To obtain the solutions of the VPE, one has to calculate the determinant of matrix (3.15). Firstly, we present four results of such a calculation for

$M + N \leq 4$ . For the sake of convenience we will use the auxiliary function  $F(X, T)$  given by the definition  $F(X, T) = \sqrt{\det M(X, T)}$ . In particular, from (3.15),

1) for  $M + N = 1$  we have

$$F = 1 + c_1 q_1; \quad (4.1)$$

2) for  $M + N = 2$  we have

$$F = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2; \quad (4.2)$$

3) for  $M + N = 3$  we have

$$F = 1 + c_1 q_1 + c_2 q_2 + c_3 q_3 + b_{12} c_1 c_2 q_1 q_2 + b_{13} c_1 c_3 q_1 q_3 + b_{23} c_2 c_3 q_2 q_3 + b_{12} b_{13} b_{23} c_1 c_2 c_3 q_1 q_2 q_3; \quad (4.3)$$

4) for  $M + N = 4$  we have

$$F = 1 + c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 + b_{12} c_1 c_2 q_1 q_2 + b_{13} c_1 c_3 q_1 q_3 + b_{14} c_1 c_4 q_1 q_4 + b_{23} c_2 c_3 q_2 q_3 + b_{24} c_2 c_4 q_2 q_4 + b_{34} c_3 c_4 q_3 q_4 + b_{12} b_{13} b_{23} c_1 c_2 c_3 q_1 q_2 q_3 + b_{12} b_{14} b_{24} c_1 c_2 c_4 q_1 q_2 q_4 + b_{13} b_{14} b_{34} c_1 c_3 c_4 q_1 q_3 q_4 + b_{23} b_{24} b_{34} c_2 c_3 c_4 q_2 q_3 q_4 + b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} c_1 c_2 c_3 c_4 q_1 q_2 q_3 q_4. \quad (4.4)$$

For  $M + N > 4$ , the explicit expression for the function  $F(X, T)$  can be obtained in a similar manner. It is reasonable to present the quantities  $c_i$ ,  $q_i$ , and  $b_{ij}$  involved in the above formulas (4.1)–(4.4) separately for three distinct cases:

1. The purely solitonic case  $(i, j) \leq M$  assumes

$$q_i = \exp(2\theta_i) \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i},$$

$$b_{ij} = \left( \frac{\xi_i - \xi_j}{\xi_i + \xi_j} \right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i \xi_j}{\xi_i^2 + \xi_j^2 + \xi_i \xi_j}, \quad b_{ij} \geq 0; \quad (4.5)$$

2. The case of purely multi-mode waves  $M < (i, j) \leq M + N$  assumes

$$q_i = \exp(2\theta_i) \quad 2\theta_i = -i\sqrt{3}\xi_i X + (i\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{i\beta_i}{2\sqrt{3}\xi_i},$$

$$b_{ij} = \left( \frac{\xi_i - \xi_j}{\xi_i + \xi_j} \right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i \xi_j}{\xi_i^2 + \xi_j^2 + \xi_i \xi_j}, \quad b_{ij} \geq 0; \quad (4.6)$$

3. The case of a combination of solitons  $(i, i') \leq M$  and multi-mode waves  $M < (j, j') \leq M + N$  assumes

$$q_i = \exp(2\theta_i) \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1} T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i},$$

$$q_j = \exp(2\theta_j) \quad 2\theta_j = -i\sqrt{3}\xi_j X + (i\sqrt{3}\xi_j)^{-1} T, \quad c_j = \frac{i\beta_j}{2\sqrt{3}\xi_j},$$



$$\begin{aligned}
b_{ii'} &= \left( \frac{\xi_i - \xi_{i'}}{\xi_i + \xi_{i'}} \right)^2 \frac{\xi_i^2 + \xi_{i'}^2 - \xi_i \xi_{i'}}{\xi_i^2 + \xi_{i'}^2 + \xi_i \xi_{i'}}, & 0 \leq b_{ii'} \leq 1, \\
b_{jj'} &= \left( \frac{\xi_j - \xi_{j'}}{\xi_j + \xi_{j'}} \right)^2 \frac{\xi_j^2 + \xi_{j'}^2 - \xi_j \xi_{j'}}{\xi_j^2 + \xi_{j'}^2 + \xi_j \xi_{j'}}, & 0 \leq b_{jj'} \leq 1, \\
b_{ij} &= \left( \frac{\xi_i + i\xi_j}{\xi_i - i\xi_j} \right)^2 \frac{\xi_i^2 - \xi_j^2 + i\xi_i \xi_j}{\xi_i^2 - \xi_j^2 - i\xi_i \xi_j}, & |b_{ij}| \equiv 1.
\end{aligned} \tag{4.7}$$

With the above found representation of the auxiliary function  $F(X, T)$  and taking into account the key relationship (3.11), we can write the explicit solution to the basic nonlinear evolution equation (1.1) in the following concise form:

$$W(X, T) = 6 \frac{\partial}{\partial X} \ln(F(X, T)) + \text{const.} \tag{4.8}$$

The function  $F$  is complex-valued in the general case because the values of  $\beta_i$  (and hence of  $c_i$ ) are complex constants.

Since we are interested only in the real solution  $W_X$  under real constants  $\xi_i$ , we need the restrictions on the constants  $c_i$  in (4.1)–(4.4).

#### 4.1. The solutions associated with the continuous spectrum

We study the multi-mode solutions for  $M = 0$  and  $N = 1, 2, 3, 4$ , while for  $N \geq 5$  all formulas can easily be obtained by means of a generalization of these examples.

##### 4.1.1. The one-mode solution

In order to obtain the one-mode solution of the VPE (1.1) we need first to calculate the  $2 \times 2$  matrix  $M(X, T)$  according to (3.15) with  $M = 0$  and  $N = 1$ . For the matrix elements  $M_{kl}(X, T)$  we have

$$\begin{aligned}
M_{11}(X, T) &= 1 - \frac{i\omega_2\beta_1}{\sqrt{3}\xi_1} \exp[-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T], \\
M_{12}(X, T) &= -\frac{\omega_3\beta_1}{2\xi_1} \exp[2\omega_3\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T], \\
M_{21}(X, T) &= \frac{\omega_2\beta_1}{2\xi_1} \exp[-2\omega_2\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T], \\
M_{22}(X, T) &= 1 - \frac{i\omega_3\beta_1}{\sqrt{3}\xi_1} \exp[-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T],
\end{aligned} \tag{4.9}$$

so that the respective determinant is

$$\det M(X, T) = \left[ 1 + c_1 \exp(-i\sqrt{3}\xi_1 X + (i\sqrt{3}\xi_1)^{-1}T) \right]^2, \quad c_1 = \frac{i\beta_1}{2\sqrt{3}\xi_1}. \tag{4.10}$$

As has been noted already, the singularity functions in the form (3.2) with  $N = 1$  give rise to a single frequency for the continuous part of the spectral data. Hence, the expression (4.10), having been substituted into the concise formula (4.8), must provide us with the one-mode solution.

The condition that  $W_X$  is real requires a restriction on the constant  $\beta_1$  (if the constant  $\xi_1$  is arbitrary but real). We have succeeded in obtaining this restriction (see Appendix A), namely that the constant  $c_1$ , which in general is the complex-valued one  $c_1 = |c_1| \exp(i\chi_1)$ , should possess the unity modulus  $|c_1| = 1$ , while the arbitrary real constant  $\chi_1$  defines an initial shift of solution  $X_1 = \chi_1/(\sqrt{3}\xi_1)$  so that

$$\det M(X, T) = \left[ 1 + \exp \left( -i\sqrt{3}\xi_1(X - X_1) + \frac{T}{i\sqrt{3}\xi_1} \right) \right]^2. \quad (4.11)$$

The final result for one mode of the continuous spectrum is the solution (4.8) with (4.11), namely,

$$W(X, T) = -3\sqrt{3}\xi_1 \tan \left( \frac{\sqrt{3}}{2}\xi_1(X - X_1) + \frac{T}{2\sqrt{3}\xi_1} \right) + \text{const.} \quad (4.12)$$

The corresponding solution for  $U = W_X$  (with  $U$  governed by (1.4)) was obtained recently by other methods, for example, by the sine-cosine method,<sup>16)</sup> the  $(G'/G)$ -expansion method,<sup>12)</sup> and the extended tanh-function method.<sup>16), 17)</sup> However, only the approach developed here and the solution in the form (3.15) enable us to study the interaction of solitons and periodic waves.

We obtain the periodic solutions even for  $N = 1$ . Let us call attention once again to the condition (2.6) which in the final result is shown below to restrict the region  $X$  for periodic solutions. At first glance it would seem that there is a contradiction between the condition (2.6) and the periodic solution. Indeed, on the one hand, the condition (2.6) demands that the solution  $W(X, T)$  should vanish as  $X \rightarrow -\infty$ ; on the other hand, the periodic solution obtained here does not satisfy the condition (2.6). Nevertheless, consideration of the details enables us to find a reasonable explanation. So, in the paper<sup>14)</sup> at the derivation of the relation (2.7) (see also (4.5) in the paper<sup>14)</sup>), the integral in (2.7) appears as a result of the integration on two sides of the boundaries between regular regions. For an understanding of this fact, the relationship (3.14) from the paper<sup>14)</sup> plays an important role. Hence, the integration in (2.7) (also as in (4.5) in the paper<sup>14)</sup>) should be carried out over the lines  $\omega_2(\xi + i\varepsilon)$  and  $-\omega_3(\xi + i\varepsilon)$  as  $\xi$  sweeps from  $-\infty$  to  $+\infty$ , where  $\varepsilon > 0$ . As a result, in the relationship (4.11) we should exchange  $\xi_1$  by  $(\xi_1 + i\varepsilon)$  and that enables us to define the solution in the form

$$W(X, T) = -i6\sqrt{3}(\xi_1 + i\varepsilon) \frac{\exp(\sqrt{3}\varepsilon X) \exp \left( -i\sqrt{3}\xi_1(X - X_1) + \frac{T}{i\sqrt{3}\xi_1} \right)}{1 + \exp(\sqrt{3}\varepsilon X) \exp \left( -i\sqrt{3}\xi_1(X - X_1) + \frac{T}{i\sqrt{3}\xi_1} \right)}, \quad (4.13)$$

which tends to constants as  $|X| \rightarrow \infty$  at arbitrary  $\varepsilon > 0$ . Thus, on the one hand, the condition (2.6) is satisfied, and, on the other hand, at small  $\varepsilon > 0$  we have a sufficiently large region over  $X$  where the solution associated with a finite  $\varepsilon > 0$  and the periodic solution associated with  $\varepsilon = 0$  are sufficiently close to each other. The region  $X$  with periodic solutions can be extended to sufficiently large, but finite,  $|X|$ . For any sequence  $\varepsilon_n \rightarrow 0$  we remain within the inverse scattering theory<sup>14)</sup> where

the condition (2.6) is not violated. Consequently, periodic solution obtained at  $\varepsilon = 0$  is to be interpreted as the solution of Eq. (1.1) which is valid on arbitrary but finite  $|X|$ .

4.1.2. *The two-mode solution*

Let us consider a two-mode solution of the VPE for which  $M = 0$  and  $N = 2$ . In this case  $M(X, T)$  is a  $4 \times 4$  matrix. According to (4.2) we find that

$$\sqrt{\det M(X, T)} = F(X, T) = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2, \tag{4.14}$$

where  $q_i$ ,  $c_i$ , and  $b_{12}$  are defined by (4.6).

Since the solution  $W_X$  should be real and the constants  $\xi_i$  are arbitrary, but real, there are restrictions on the constants  $c_i = |c_i| \exp(i\chi_i)$ . The real constants  $\chi_i$  define the initial shifts of solutions  $X_i = \chi_i/(\sqrt{3}\xi_i)$ . The analysis in considerable detail shows (see Appendix A) that the relations  $|c_1| = |c_2| = 1/\sqrt{b_{12}}$  are the sufficient conditions in order that  $W_X$  be real. Thus, the interaction of two periodic waves for the VPE is described by the relationship (4.8) with

$$F(X, T) = 1 + \frac{1}{\sqrt{b_{12}}}q_1 + \frac{1}{\sqrt{b_{12}}}q_2 + q_1q_2, \tag{4.15}$$

where  $b_{12}$  is as in (4.6), and the quantities  $q_i$  now contain the phaseshifts  $X_i = \chi_i/(\sqrt{3}\xi_i)$  as follows:

$$q_i = \exp(i2\theta_i), \quad 2\theta_i = -\sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T. \tag{4.16}$$

In explicit form, the two-mode solution is as follows:

$$W(X, T) = -3\sqrt{3} \frac{(\xi_1 + \xi_2) \sin(\theta_1 + \theta_2) + b_{12}^{-1/2}(\xi_1 - \xi_2) \sin(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2) + b_{12}^{-1/2} \cos(\theta_1 - \theta_2)}. \tag{4.17}$$

4.1.3. *The three-mode solution*

For  $N = 3$  and  $M = 0$  in the relationship

$$F(X, T) = 1 + c_1q_1 + c_2q_2 + c_3q_3 + c_1c_2b_{12}q_1q_2 + c_1c_3b_{13}q_1q_3 + c_2c_3b_{23}q_2q_3 + c_1c_2c_3b_{12}b_{13}b_{23}q_1q_2q_3 \tag{4.18}$$

obtained from (3.15) (see, also (4.3)) with  $q_i$ ,  $c_i$ , and  $b_{12}$  as in (4.6), we write  $c_i = |c_i| \exp(i\chi_i)$ . Then the arguments  $\chi_i$  determine the initial phaseshifts of modes  $X_i = \chi_i/(\sqrt{3}\xi_i)$ . As is proved in Appendix A, the conditions on the constants  $c_i$  (or the same on  $\beta_i$ ) are

$$|c_1| = 1/\sqrt{b_{12}b_{13}}, \quad |c_2| = 1/\sqrt{b_{12}b_{23}}, \quad |c_3| = 1/\sqrt{b_{13}b_{23}}. \tag{4.19}$$

Hence, the three-mode solution is the relation (4.8) with

$$F(X, T) = 1 + \frac{1}{\sqrt{b_{12}b_{13}}}(q_1 + q_2q_3) + \frac{1}{\sqrt{b_{12}b_{23}}}(q_2 + q_1q_3) + \frac{1}{\sqrt{b_{13}b_{23}}}(q_3 + q_1q_2) + q_1q_2q_3. \tag{4.20}$$

Here the phaseshifts  $X_i$  are taken into account in  $q_i$  by way of (4.16).

4.1.4. *The four-mode solution*

For  $N = 4$  and  $M = 0$  the restrictions have the form (see Appendix A)

$$|c_i| = \prod_{\substack{j=1 \\ j \neq i}}^4 b_{ij}^{-\frac{1}{2}}, \quad 0 \leq b_{ij} = b_{ji} \leq 1, \quad i = 1, 2, 3, 4. \tag{4.21}$$

The function  $F$  for a real solution (4.8) is as follows:

$$\begin{aligned} F(X, T) = & 1 + \frac{1}{\sqrt{b_{12}b_{13}b_{14}}}(q_1 + q_2q_3q_4) + \frac{1}{\sqrt{b_{12}b_{23}b_{24}}}(q_2 + q_1q_3q_4) \\ & + \frac{1}{\sqrt{b_{13}b_{23}b_{34}}}(q_3 + q_1q_2q_4) + \frac{1}{\sqrt{b_{14}b_{24}b_{34}}}(q_4 + q_1q_2q_3) \\ & + \frac{1}{\sqrt{b_{13}b_{14}b_{23}b_{24}}}(q_1q_2 + q_3q_4) + \frac{1}{\sqrt{b_{12}b_{14}b_{23}b_{34}}}(q_1q_3 + q_2q_4) \\ & + \frac{1}{\sqrt{b_{12}b_{13}b_{24}b_{34}}}(q_1q_4 + q_2q_3) + q_1q_2q_3q_4. \end{aligned} \tag{4.22}$$

As before, the  $b_{ij}$  and  $q_i$  are defined by (4.6) and (4.16), respectively.

4.2. *The solutions associated with the bound state spectrum*

The features of the solutions associated with bound state spectrum can be shown by considering the two-soliton solution for which  $M = 2$ ,  $N = 0$ . The solution (4.8) can be obtained through (4.2), namely

$$F(X, T) = 1 + c_1q_1 + c_2q_2 + b_{12}c_1c_2q_1q_2 \tag{4.23}$$

with (4.5), namely

$$\begin{aligned} q_i = \exp(2\theta_i) \quad 2\theta_i = \sqrt{3}\xi_i X - (\sqrt{3}\xi_i)^{-1}T, \quad c_i = \frac{\beta_i}{2\sqrt{3}\xi_i}, \\ b_{ij} = \left( \frac{\xi_i - \xi_j}{\xi_i + \xi_j} \right)^2 \frac{\xi_i^2 + \xi_j^2 - \xi_i\xi_j}{\xi_i^2 + \xi_j^2 + \xi_i\xi_j}, \quad b_{ij} \geq 0. \end{aligned} \tag{4.24}$$

In Appendix B it is proved that the constants  $c_i$  can be only real ones. Moreover, the signs of  $\alpha_i = c_i/|c_i|$  can independently take the values  $\pm 1$ , i.e. we have four variants, namely  $\alpha_1 = \alpha_2 = 1$ ,  $\alpha_1 = \alpha_2 = -1$ ,  $\alpha_1 = -\alpha_2 = 1$  and  $\alpha_1 = -\alpha_2 = -1$ . Note that in the paper<sup>2)</sup> only the first two variants are observed. The standard soliton solution for which  $\alpha_1 = \alpha_2 = 1$  and the singular soliton solutions for which  $\alpha_1 = \alpha_2 = -1$ ,  $\alpha_1 = -\alpha_2 = 1$  and  $\alpha_1 = -\alpha_2 = -1$ , are obtained by means of the relation (4.8)

$$U(X, T) = W(X, T)_X = 6 \frac{\partial^2}{\partial X^2} \ln(F) = 6 \frac{\partial^2}{\partial X^2} \ln(G_i), \tag{4.25}$$

where the  $G_i$  are defined by (B.6)–(B.9).

The forms (B.3), (B.6)–(B.9) for  $F$  are more preferable, since we see that the solution is dependent on two combinations of the spectral parameters  $\xi_1 + \xi_2$  and  $\xi_1 - \xi_2$ , but not three values  $\xi_1$ ,  $\xi_2$ , and  $\xi_1 + \xi_2$  as it may appear from the relation (4.23).

For  $N \geq 3$  we give the conditions without proof. All the constants  $c_i$  are to be real and the signs of  $\alpha_i = c_i/|c_i|$  can equal to  $\pm 1$  independently of each other.

### 4.3. Real soliton and multi-mode solutions of the VPE

In this subsection we will consider the general case, when both the bound state spectrum and the continuous spectrum are taken into account in the associated spectral problem. We will find the conditions on  $c_i$  for real solutions of the VPE. To obtain the solution, we need to know the function  $F$  (see (4.1)–(4.4)).

Let the indexes  $i$  and  $i'$  be related to the values involved in the bound state spectrum for which  $(i, i') \leq M$ , while the indexes  $j$  and  $j'$  are related to the values involved in the continuous part of the spectral data for which  $M < (j, j') \leq M + N$ .

#### 4.3.1. The interaction of a soliton with one-mode wave

The interaction of a standard soliton with periodic one-mode wave can be described by means of the relations (4.2)

$$F(X, T) = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2 \tag{4.26}$$

with  $q_i$  and  $b_{12}$  as in (4.7), namely

$$\begin{aligned} q_1 &= \exp(\sqrt{3}\xi_1 X - (\sqrt{3}\xi_1)^{-1}T), & c_1 &= \frac{\beta_1}{2\sqrt{3}\xi_1}, \\ q_2 &= \exp(-i\sqrt{3}\xi_2 X + (i\sqrt{3}\xi_2)^{-1}T), & c_2 &= \frac{i\beta_2}{2\sqrt{3}\xi_2}, \\ b_{12} &= \left( \frac{\xi_1 + i\xi_2}{\xi_1 - i\xi_2} \right)^2 \frac{\xi_1^2 - \xi_2^2 + i\xi_1\xi_2}{\xi_1^2 - \xi_2^2 - i\xi_1\xi_2}, & |b_{12}| &\equiv 1. \end{aligned} \tag{4.27}$$

First, we emphasize that the soliton and one-mode wave (4.12) propagate in opposite directions. The soliton propagates in the positive direction of the  $X$ -axis, while the one-mode wave (4.12) propagates in the negative direction of the  $X$ -axis.

Here we restrict ourselves to the simplest case  $b_{12}c_1c_2 = 1$  that describes the interaction of a standard soliton with a one-mode wave. As follows immediately from Appendix C, for real solutions (4.8),

$$W(X, T) = 6 \frac{\partial}{\partial X} \ln(F(X, T)) + \text{const},$$

where  $F(X, T)$  is

$$F(X, T) = 1 + \frac{1}{\sqrt{b_{12}}}q_1 + \frac{1}{\sqrt{b_{12}}}q_2 + q_1q_2. \tag{4.28}$$

There is an exceptional case at  $\xi_1 = \xi_2$ . Then we have  $b_{12} = 1$ , and  $F = (1 + q_1)(1 + q_2)$ . Consequently, the solution (4.8) is reduced to the relation

$$\begin{aligned} W = W_1 + W_2 &= 3\sqrt{3}\xi_1 \tanh \left( \frac{\sqrt{3}}{2}\xi_1(X - X_1) - \frac{T}{2\sqrt{3}\xi_1} \right) \\ &\quad - 3\sqrt{3}\xi_1 \tan \left( \frac{\sqrt{3}}{2}\xi_1(X - X_0) + \frac{T}{2\sqrt{3}\xi_1} \right) + \text{const}. \end{aligned} \tag{4.29}$$

Here  $W_1$  is the one-soliton solution and  $W_2$  is the solution (4.12) associated with one mode in the continuous part of the spectral data. The relationship  $W = W_1 + W_2$

is easily verified also by direct substitution into Eq. (1·1). The two waves  $W_1$  and  $W_2$  propagate in different directions with the same speed without change of wave profile and phaseshift. In other words, only in the case  $\xi_1 = \xi_2$  is there a simple superposition of the solutions  $W_1$  and  $W_2$ . It is obvious that interactions of the two solitons with a one-mode wave and/or of the two-mode solution with one soliton do not satisfy this form of the interaction.

#### 4.3.2. Real solutions for $M$ solitons and the $N$ -mode wave

The interaction of  $M$  solitons and the  $N$ -mode wave (4·8) can be obtained by means of the function  $F(X, T)$  with restrictions (C·7) given in Appendix C, namely

$$c_i = \pm 1 \left/ \sqrt{\prod_{\substack{j=1 \\ j \neq i}}^{M+N} b_{ij}} \right., \quad b_{ij} = b_{ji}, \quad i = 1, \dots, M + N, \quad (4\cdot30)$$

and with the retention of the phaseshifts  $X_i$  in the quantities  $q_i$  (C·2). The signs for  $c_i$  in (4·30) can be chosen independently of each other. If the index  $i$  in (4·30) is connected with the continuous part of the spectral data ( $M < i \leq M + N$ ), then the solutions generated by ‘plus’ and ‘minus’ signs in (4·30) are different only in the phaseshifts. However, for the index  $i$  from the bound state spectrum ( $i \leq M$ ), the solutions have different forms of function dependencies. Here it is relevant to remember that there are standard soliton solutions and singular soliton solutions generated by different signs in the constants  $c_i$  (4·30).

The solution will contain  $(M + N)$  real constants  $\xi_i$  for determining the values  $b_{ij}$  and  $(M + N)$  real constants  $X_i$  to define the phaseshifts.

## §5. Conclusion

The procedure for finding the solutions of the Vakhnenko-Parkes equation by means of the inverse scattering method is described. Both the bound state spectrum and the continuous spectrum are taken into account in the associated eigenvalue problem. The special form of the singularity functions enables us to obtain the multi-mode solutions. Sufficient conditions have been proved in order that the solutions become real functions. Finally we studied the interaction of solitons and the multi-mode wave.

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## Appendix A

— The Conditions on the Constants  $c_i$  for Multi-Mode Waves —

In this appendix we will prove the conditions on the constants  $c_i = |c_i| \exp(i\chi_i)$

for solutions associated with the continuous part of the spectral data only. We use the case  $N = 4$  as an example to prove the restrictions on the constants, at which the solution  $W_X(X, T)$  is real. The auxiliary function  $F(X, T) = \sqrt{\det M(X, T)}$  for finding the solution is (4.4), namely

$$\begin{aligned}
 F(X, T) = & 1 + c_1q_1 + c_2q_2 + c_3q_3 + c_4q_4 + c_1c_2b_{12}q_1q_2 + c_1c_3b_{13}q_1q_3 \\
 & + c_1c_4b_{14}q_1q_4 + c_2c_3b_{23}q_2q_3 + c_2c_4b_{24}q_2q_4 + c_3c_4b_{34}q_3q_4 \\
 & + c_1c_2c_3b_{12}b_{13}b_{23}q_1q_2q_3 + c_1c_2c_4b_{12}b_{14}b_{24}q_1q_2q_4 \\
 & + c_1c_3c_4b_{13}b_{14}b_{34}q_1q_3q_4 + c_2c_3c_4b_{23}b_{24}b_{34}q_2q_3q_4 \\
 & + c_1c_2c_3c_4b_{12}b_{13}b_{14}b_{23}b_{24}b_{34}q_1q_2q_3q_4.
 \end{aligned} \tag{A.1}$$

Here we redefine the values  $c_i$  in such a way that  $c_i = |c_i|$ , since the arguments  $\chi_i$  can always be introduced into the variables  $q_i = \exp(i2\theta_i)$  with  $2\theta_i = -\sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T$  and  $X_i = \chi_i/(\sqrt{3}\xi_i)$  serving as the shifts of solutions. The solution then has the form (4.8)

$$W(X, T) = 6 \frac{\partial}{\partial X} \ln(F(X, T)) + \text{const.} \tag{A.2}$$

The function  $F$  is complex-valued, i.e.

$$\begin{aligned}
 F &= F_{Re} + iF_{Im} = |F| \exp(i\chi_F), \quad F_{Re} = \text{Re}(F), \quad F_{Im} = \text{Im}(F), \\
 \tan(\chi_F) &= F_{Im}/F_{Re},
 \end{aligned} \tag{A.3}$$

hence,

$$W(X, T)/6 = \frac{\partial}{\partial X} \ln(|F|) + i \frac{\partial \chi_F}{\partial X} + \text{const.} \tag{A.4}$$

If we succeed in making  $\partial^2 \chi_F / \partial X^2 \equiv 0$  by the choice of the constants  $c_i$ , then the solution  $W_X(X, T)$  will be a real function.

Let us write  $F_{Im}$  and  $F_{Re}$  in explicit forms, namely

$$\begin{aligned}
 F_{Im} = & c_1 \sin(2\theta_1) + c_2 \sin(2\theta_2) + c_3 \sin(2\theta_3) + c_4 \sin(2\theta_4) \\
 & + c_1c_2b_{12} \sin[2(\theta_1 + \theta_2)] + c_1c_3b_{13} \sin[2(\theta_1 + \theta_3)] \\
 & + c_1c_4b_{14} \sin[2(\theta_1 + \theta_4)] + c_2c_3b_{23} \sin[2(\theta_2 + \theta_3)] \\
 & + c_2c_4b_{24} \sin[2(\theta_2 + \theta_4)] + c_3c_4b_{34} \sin[2(\theta_3 + \theta_4)] \\
 & + c_1c_2c_3b_{12}b_{13}b_{23} \sin[2(\theta_1 + \theta_2 + \theta_3)] \\
 & + c_1c_2c_4b_{12}b_{14}b_{24} \sin[2(\theta_1 + \theta_2 + \theta_4)] \\
 & + c_1c_3c_4b_{13}b_{14}b_{34} \sin[2(\theta_1 + \theta_3 + \theta_4)] \\
 & + c_2c_3c_4b_{23}b_{24}b_{34} \sin[2(\theta_2 + \theta_3 + \theta_4)] \\
 & + c_1c_2c_3c_4b_{12}b_{13}b_{14}b_{23}b_{24}b_{34} \sin[2(\theta_1 + \theta_2 + \theta_3 + \theta_4)],
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 F_{Re} = & 1 + c_1 \cos(2\theta_1) + c_2 \cos(2\theta_2) + c_3 \cos(2\theta_3) + c_4 \cos(2\theta_4) \\
 & + c_1c_2b_{12} \cos[2(\theta_1 + \theta_2)] + c_1c_3b_{13} \cos[2(\theta_1 + \theta_3)] \\
 & + c_1c_4b_{14} \cos[2(\theta_1 + \theta_4)] + c_2c_3b_{23} \cos[2(\theta_2 + \theta_3)]
 \end{aligned}$$

$$\begin{aligned}
 &+c_2c_4b_{24} \cos[2(\theta_2 + \theta_4)] + c_3c_4b_{34} \cos[2(\theta_3 + \theta_4)] \\
 &+c_1c_2c_3b_{12}b_{13}b_{23} \cos[2(\theta_1 + \theta_2 + \theta_3)] \\
 &+c_1c_2c_4b_{12}b_{14}b_{24} \cos[2(\theta_1 + \theta_2 + \theta_4)] \\
 &+c_1c_3c_4b_{13}b_{14}b_{34} \cos[2(\theta_1 + \theta_3 + \theta_4)] \\
 &+c_2c_3c_4b_{23}b_{24}b_{34} \cos[2(\theta_2 + \theta_3 + \theta_4)] \\
 &+c_1c_2c_3c_4b_{12}b_{13}b_{14}b_{23}b_{24}b_{34} \cos[2(\theta_1 + \theta_2 + \theta_3 + \theta_4)].
 \end{aligned} \tag{A.6}$$

Let us try to present  $F_{Im}$  and  $F_{Re}$  in the forms

$$F_{Im} = 2G \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \tag{A.7}$$

and

$$F_{Re} = 2G \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4), \tag{A.8}$$

where  $G$  is the same in both above formulas (A.7) and (A.8). This can be done, if the following conditions are satisfied:

$$\begin{aligned}
 c_1 &= c_2c_3c_4b_{23}b_{24}b_{34}, & c_2 &= c_1c_3c_4b_{13}b_{14}b_{34}, & c_3 &= c_1c_2c_4b_{12}b_{14}b_{24}, \\
 c_4 &= c_1c_2c_3b_{12}b_{13}b_{23}, & c_1c_2b_{12} &= c_3c_4b_{34}, & c_1c_3b_{13} &= c_2c_4b_{24}, \\
 c_1c_4b_{14} &= c_2c_3b_{23}, & c_1c_2c_3c_4b_{12}b_{13}b_{14}b_{23}b_{24}b_{34} &= 1.
 \end{aligned} \tag{A.9}$$

It turns out that all these relations are valid when

$$c_1 = \frac{1}{\sqrt{b_{12}b_{13}b_{14}}}, \quad c_2 = \frac{1}{\sqrt{b_{12}b_{23}b_{24}}}, \quad c_3 = \frac{1}{\sqrt{b_{13}b_{23}b_{34}}}, \quad c_4 = \frac{1}{\sqrt{b_{14}b_{24}b_{34}}}. \tag{A.10}$$

At these conditions (A.10) the expression for  $G$  reads as follows:

$$\begin{aligned}
 G &= \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) + \frac{1}{\sqrt{b_{12}b_{13}b_{14}}} \cos(\theta_1 - \theta_2 - \theta_3 - \theta_4) \\
 &+ \frac{1}{\sqrt{b_{12}b_{23}b_{24}}} \cos(\theta_2 - \theta_1 - \theta_3 - \theta_4) + \frac{1}{\sqrt{b_{13}b_{23}b_{34}}} \cos(\theta_3 - \theta_1 - \theta_2 - \theta_4) \\
 &+ \frac{1}{\sqrt{b_{14}b_{24}b_{34}}} \cos(\theta_4 - \theta_1 - \theta_2 - \theta_3) + \frac{1}{\sqrt{b_{13}b_{14}b_{23}b_{24}}} \cos(\theta_1 + \theta_2 - \theta_3 - \theta_4) \\
 &+ \frac{1}{\sqrt{b_{12}b_{14}b_{23}b_{34}}} \cos(\theta_1 + \theta_3 - \theta_2 - \theta_4) + \frac{1}{\sqrt{b_{12}b_{13}b_{24}b_{34}}} \cos(\theta_1 + \theta_4 - \theta_2 - \theta_3).
 \end{aligned} \tag{A.11}$$

Now it is readily seen from (A.3) that

$$\chi_F = \theta_1 + \theta_2 + \theta_3 + \theta_4 \tag{A.12}$$

and as consequence we have

$$\frac{\partial^2 \chi_F}{\partial X^2} = \frac{\partial^2 \chi_F}{\partial X \partial T} = 0. \tag{A.13}$$

Hence, as follows from (A.4), the four-mode solution of the VPE can be reduced to real form with four real constants  $X_i$  and four real constants  $\xi_i$  (see (4.22)).



Without proof here we give the following conditions on the constants  $c_i$  that ensure the real  $N$ -mode solution of the VPE:

$$|c_i| = \prod_{\substack{j=1 \\ j \neq i}}^N b_{ij}^{-\frac{1}{2}}, \quad b_{ij} = b_{ji}, \quad i = 1, \dots, N, \tag{A.14}$$

where the  $N$  constants  $\xi_i$  determine the values  $b_{ij}$  and the  $N$  constants  $X_i$  define the phaseshifts for each mode. Note that these relations (A.14) are sufficient conditions, but not necessary ones.

### Appendix B

#### — The Conditions on the Constants $c_i$ under the Interaction of Two Solitons —

Here we consider the conditions on signs for the constants  $c_i$  under the interaction of two solitons ( $M = 2, N = 0$ ). We start with the relationship (4.2) and (4.5)

$$F = 1 + c_1 q_1 + c_2 q_2 + b_{12} c_1 c_2 q_1 q_2. \tag{B.1}$$

Let us present the constants  $c_i$  in the form

$$c_i = \alpha_i |c_i| \exp(i\chi_i) = b_{12}^{-1/2} \exp\left(-\sqrt{3}\xi_i X_i + i\sigma_i\right), \quad \sigma_i = \chi_i + \pi(1 - \alpha_i)/2. \tag{B.2}$$

All new constants  $\chi_i$  and  $X_i = -\ln(|c_i \sqrt{b_{12}}|)/(\sqrt{3}\xi_i)$  are real. We assume that  $-\pi/2 < \chi_i \leq \pi/2$ , then the values  $\alpha_i$  retain the signs of the constants  $\text{Re}(c_i)$ , i.e.  $\alpha_i = \text{Re}(c_i)/|\text{Re}(c_i)|$ . It is convenient for analyzing to rewrite (B.1) (the same as (4.2)) in the form

$$F = 2 \exp\left(\theta_1 + \theta_2 + \frac{i}{2}(\sigma_1 + \sigma_2)\right) G \tag{B.3}$$

with

$$G = \cosh\left(\theta_1 + \theta_2 + \frac{i}{2}(\sigma_1 + \sigma_2)\right) + b_{12}^{-1/2} \cosh\left(\theta_1 - \theta_2 + \frac{i}{2}(\sigma_1 - \sigma_2)\right),$$

$$2\theta_i = \sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T. \tag{B.4}$$

It is easily seen that only  $G$  defines the solution, since  $\frac{\partial^2}{\partial X^2} \ln(F) = \frac{\partial^2}{\partial X^2} \ln(G)$ , while the conditions that the function  $G$  is real are as follows:

$$\chi_i = 0, \quad \sigma_i + \sigma_2 = 2\pi k_1, \quad \sigma_i - \sigma_2 = 2\pi k_2 \tag{B.5}$$

with  $k_i = 0, 1$ . These restrictions (B.5) lead to the requirements  $\alpha_1 = \pm 1, \alpha_2 = \pm 1$ , independently of each other, and  $\chi_i = 0$ . Then the function  $F$  has the following forms:

1) for  $\alpha_1 = \alpha_2 = 1$ ,

$$F = 2 \exp(\theta_1 + \theta_2) G_1, \quad G_1 = \cosh(\theta_1 + \theta_2) + b_{12}^{-1/2} \cosh(\theta_1 - \theta_2); \tag{B.6}$$

2) for  $\alpha_1 = \alpha_2 = -1$ ,

$$F = 2 \exp(\theta_1 + \theta_2) G_2, \quad G_2 = \cosh(\theta_1 + \theta_2) - b_{12}^{-1/2} \cosh(\theta_1 - \theta_2); \quad (\text{B}\cdot 7)$$

3) for  $\alpha_1 = -\alpha_2 = 1$ ,

$$F = 2 \exp(\theta_1 + \theta_2) G_3, \quad G_3 = -\sinh(\theta_1 + \theta_2) + b_{12}^{-1/2} \sinh(\theta_1 - \theta_2); \quad (\text{B}\cdot 8)$$

4) for  $\alpha_1 = -\alpha_2 = -1$ ,

$$F = 2 \exp(\theta_1 + \theta_2) G_4, \quad G_4 = -\sinh(\theta_1 + \theta_2) - b_{12}^{-1/2} \sinh(\theta_1 - \theta_2). \quad (\text{B}\cdot 9)$$

Hence, the standard soliton solution that follows from (B·6) and the singular soliton solutions that follow from (B·7)–(B·9) are the real functions

$$U(X, T) = W_X(X, T) = 6 \frac{\partial^2}{\partial X^2} \ln(G_i). \quad (\text{B}\cdot 10)$$

Now we rewrite the restrictions in somewhat different form. By retaining the values of the phaseshifts  $X_i$  in the quantities  $q_i$ , we require

$$c_1 = \pm \sqrt{b_{12}}, \quad c_2 = \pm \sqrt{b_{12}}, \quad (\text{B}\cdot 11)$$

where the signs are independent of each other. Note that for this case there are two arbitrary real constants  $\xi_i$ , and two arbitrary real constants  $X_i$  ( $i = 1, 2$ ).

The notation in (B·6)–(B·9) shows that the solution is defined by two combinations of the spectral parameters, namely  $\xi_1 + \xi_2$  and  $\xi_1 - \xi_2$ , but not three values  $\xi_1$ ,  $\xi_2$ ,  $\xi_1 + \xi_2$  as it may appear from (B·1).

The foregoing proof points to a way for finding the restrictions for any  $M$  with  $N = 0$ . Here it should be underlined that only at real  $c_i$  with any sign of  $\alpha_i = c_i/|c_i|$ , the soliton (or singular soliton) solutions are determined by a real function. The conditions on the constants  $c_i$  are as follows:

$$c_i = \pm 1 \left/ \sqrt{\prod_{\substack{j=1 \\ j \neq i}}^M b_{12}} \right., \quad i = 1, \dots, M, \quad (\text{B}\cdot 12)$$

with the retention of the phaseshifts  $X_i$  in the quantities  $q_i$ . The signs for  $c_i$  are independent of each other. The solution will contain the  $M$  real constants  $\xi_i$  for determining the values  $b_{ij}$  and the  $M$  real constants  $X_i$  to define the phaseshifts.

### Appendix C

#### — The Restrictions on the Constants $c_i$ in the General Case —

In this appendix we will obtain the restrictions on the constants  $c_i$  for real solutions, in the general case, taking into account the spectral data from both the bound state spectrum and the continuous spectrum. All features are inherent in the

case  $M + N = 4$  considered here as an example. To find the solution by means of the inverse scattering method, one needs to know the function (4.4)

$$\begin{aligned}
 F = & 1 + c_1q_1 + c_2q_2 + c_3q_3 + c_4q_4 + b_{12}c_1c_2q_1q_2 + b_{13}c_1c_3q_1q_3 \\
 & + b_{14}c_1c_4q_1q_4 + b_{23}c_2c_3q_2q_3 + b_{24}c_2c_4q_2q_4 + b_{34}c_3c_4q_3q_4 \\
 & + b_{12}b_{13}b_{23}c_1c_2c_3q_1q_2q_3 + b_{12}b_{14}b_{24}c_1c_2c_4q_1q_2q_4 \\
 & + b_{13}b_{14}b_{34}c_1c_3c_4q_1q_3q_4 + b_{23}b_{24}b_{34}c_2c_3c_4q_2q_3q_4 \\
 & + b_{12}b_{13}b_{14}b_{23}b_{24}b_{34}c_1c_2c_3c_4q_1q_2q_3q_4.
 \end{aligned} \tag{C.1}$$

For convenience we rewrite the variables  $q_i$  in the somewhat different form

$$\begin{aligned}
 q_i = \exp(2\theta_i), \quad q_j = \exp(i2\theta_j), \quad 2\theta_i = \sqrt{3}\xi_i(X - X_i) - (\sqrt{3}\xi_i)^{-1}T, \\
 2\theta_j = -\sqrt{3}\xi_j(X - X_j) - (\sqrt{3}\xi_j)^{-1}T.
 \end{aligned} \tag{C.2}$$

The phaseshifts  $X_i$  are the arbitrary real constants. The values  $b_{ij}$  in (C.1) are as in (4.7)

$$\begin{aligned}
 b_{ii'} &= \left( \frac{\xi_i - \xi_{i'}}{\xi_i + \xi_{i'}} \right)^2 \frac{\xi_i^2 + \xi_{i'}^2 - \xi_i\xi_{i'}}{\xi_i^2 + \xi_{i'}^2 + \xi_i\xi_{i'}}, \quad 0 \leq b_{ii'} \leq 1, \\
 b_{jj'} &= \left( \frac{\xi_j - \xi_{j'}}{\xi_j + \xi_{j'}} \right)^2 \frac{\xi_j^2 + \xi_{j'}^2 - \xi_j\xi_{j'}}{\xi_j^2 + \xi_{j'}^2 + \xi_j\xi_{j'}}, \quad 0 \leq b_{jj'} \leq 1, \\
 b_{ij} &= \left( \frac{\xi_i + i\xi_j}{\xi_i - i\xi_j} \right)^2 \frac{\xi_i^2 - \xi_j^2 + i\xi_i\xi_j}{\xi_i^2 - \xi_j^2 - i\xi_i\xi_j}, \quad |b_{ij}| \equiv 1,
 \end{aligned} \tag{C.3}$$

where  $(i, i') \leq M$ , and  $M < (j, j') \leq M + N$ . Note that  $b_{ii'}$  and  $b_{jj'}$  are real values, and  $b_{ij}^* = 1/b_{ij}$ .

Without loss of generality, we will consider one set of values  $M$  and  $N$ , for example  $M = 1$  and  $N = 3$ . Now we will show that the restrictions (A.10)

$$\begin{aligned}
 c_1 = \pm 1/\sqrt{b_{12}b_{13}b_{14}}, \quad c_2 = \pm 1/\sqrt{b_{12}b_{23}b_{24}}, \\
 c_3 = \pm 1/\sqrt{b_{13}b_{23}b_{34}}, \quad c_4 = \pm 1/\sqrt{b_{14}b_{24}b_{34}}
 \end{aligned} \tag{C.4}$$

(with  $b_{ij}$  determined by (C.3)) are sufficient in order to obtain the real solutions.

For definiteness, we assume that  $\sqrt{b_{ij}}$  is a root of an equation  $x^2 = b_{ij}$  with  $-\pi/2 < \arg \sqrt{b_{ij}} \leq \pi/2$ . Let us rewrite the relations (C.4) in the form  $c_i = \alpha_i / \prod_{\substack{j=1 \\ j \neq i}}^4 \sqrt{b_{ij}}$ , where  $\alpha_i = \pm 1$ . It is evident that we can always attain  $\alpha_2 = \alpha_3 = \alpha_4 = 1$  by choosing the phaseshifts  $X_2, X_3$ , and  $X_4$ , while we need to consider the two cases  $\alpha_1 = \pm 1$ . By defining  $\sigma = (1 - \alpha_1)/2$ , we can rewrite the auxiliary function  $F$  from (C.1) in the form

$$F(X, T) = 2Ge^{i\pi\sigma} (b_{12}b_{13}b_{14})^{-1/4} \exp(\theta_1 + i\pi\sigma/2 + i\theta_2 + i\theta_3 + i\theta_4),$$

$$Ge^{i\pi\sigma} = \left[ (b_{12}b_{13}b_{14})^{1/4} \cos(-i\theta_1 + \pi\sigma/2 + \theta_2 + \theta_3 + \theta_4) \right]$$

$$\begin{aligned}
& + (b_{12}b_{13}b_{14})^{-1/4} \cos(-i\theta_1 + \pi\sigma/2 - \theta_2 - \theta_3 - \theta_4) \Big] \\
& + (b_{23}b_{24})^{-1/2} \left[ (b_{13}b_{14}/b_{12})^{1/4} \cos(i\theta_1 - \pi\sigma/2 + \theta_2 - \theta_3 - \theta_4) \right. \\
& + (b_{13}b_{14}/b_{12})^{-1/4} \cos(-i\theta_1 + \pi\sigma/2 + \theta_2 - \theta_3 - \theta_4) \Big] \\
& + (b_{23}b_{34})^{-1/2} \left[ (b_{12}b_{14}/b_{13})^{1/4} \cos(i\theta_1 - \pi\sigma/2 + \theta_3 - \theta_2 - \theta_4) \right. \\
& + (b_{12}b_{14}/b_{13})^{-1/4} \cos(-i\theta_1 + \pi\sigma/2 + \theta_3 - \theta_2 - \theta_4) \Big] \\
& + (b_{24}b_{34})^{-1/2} \left[ (b_{12}b_{13}/b_{14})^{1/4} \cos(i\theta_1 - \pi\sigma/2 + \theta_4 - \theta_2 - \theta_3) \right. \\
& + (b_{12}b_{13}/b_{14})^{-1/4} \cos(-i\theta_1 + \pi\sigma/2 + \theta_4 - \theta_2 - \theta_3) \Big]. \tag{C.5}
\end{aligned}$$

Since  $b_{23}$ ,  $b_{24}$ , and  $b_{34}$  are real, and  $b_{1j}^* = 1/b_{1j}$  for  $j = 2, 3, 4$ , it is evident that  $G^* = G$ , i.e. the variable  $G$  in the solution is a real-valued function. Hence, the solution of the VPE

$$U(X, T) = W_X(X, T) = 6 \frac{\partial^2}{\partial X^2} \ln(F) = 6 \frac{\partial^2}{\partial X^2} \ln(G) \tag{C.6}$$

represents a real quantity.

Using this example, one can prove without difficulty that the procedure considered above can be extended to any  $M$  and  $N$  with restrictions (see also (A.14), (B.12), and (C.4))

$$c_i = \pm 1 \Big/ \sqrt{\prod_{\substack{j=1 \\ j \neq i}}^{M+N} b_{ij}}, \quad b_{ij} = b_{ji}, \quad i = 1, \dots, M + N, \tag{C.7}$$

while the quantities  $q_i$  retain the phaseshifts  $X_i$  (see (C.2)). The signs in (C.7) can be chosen independently of each other. For interaction of  $M$  solitons and the  $N$ -mode wave there are  $(M + N)$  real constants  $\xi_i$  and  $(M + N)$  real constants  $X_i$ .

Note that the restrictions (C.7) are sufficient conditions in order that the solution of the VPE becomes real.

### References

- 1) R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).
- 2) A. M. Wazwaz, *Physica Scripta* **82** (2010), 065006.
- 3) M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform* (SIAM Press, Philadelphia, 1981).
- 4) *Bäcklund transformations, the inverse scattering method, solitons, and their applications*, ed. R. M. Miura (Springer, New York, 1976).
- 5) S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of solitons: The inverse scattering methods* (Plenum Publishing Corporation, New York-London, 1984).
- 6) V. A. Vakhnenko, *J. of Phys. A* **25** (1992), 4181.
- 7) E. J. Parkes, *J. of Phys. A* **26** (1993), 6469.
- 8) V. O. Vakhnenko, *J. Math. Phys.* **40** (1999), 2011.
- 9) V. O. Vakhnenko and E. J. Parkes, *Nonlinearity* **11** (1998), 1457.

- 10) A. J. Morrison, E. J. Parkes and V. O. Vakhnenko, *Nonlinearity* **12** (1999), 1427.
- 11) P. G. Estévez, *Theor. Math. Phys.* **159** (2009), 763.
- 12) R. Abazari, *Computers & Fluids* **39** (2010), 1957.
- 13) V. O. Vakhnenko and E. J. Parkes, *Chaos, Solitons & Fractals* **13** (2002), 1819.
- 14) P. J. Caudrey, *Physica D* **6** (1982), 51.
- 15) D. J. Kaup, *Stud. Appl. Math.* **62** (1980), 189.
- 16) E. Yusufoglu and A. Bekir, *Chaos, Solitons & Fractals* **38** (2008), 1126.
- 17) E. J. Parkes, *Computers & Fluids* **42** (2011), 108.