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A novel nonlinear evolution equation and its Bäcklund transformation

(Presented by Corresponding Member of the NAS of Ukraine A. G. Zagorodnii)

Для рівняння $(u_t + uu_x)_x + u = 0$, що записане в дещо інших координатах, виведено перетворення Беклунда як у білінійному вигляді, так і у звичайному. Формулюється обернена задача розсіювання. Обернений метод розсіювання утримує задачу третього порядку на власні значення. Для прикладу наводиться односолітонний розв'язок рівняння Вахненка, одержаний методом оберненої задачі розсіювання.

1. Introduction. This paper deals with the nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0 \tag{1}$$

which was first presented by Vakhnenko in [1] to describe high frequency waves in a relaxing medium [2]. Hereafter, as was initiated in [3], this equation (1) is referred to as the Vakhnenko equation (VE). A remarkable feature of the VE is that it has a soliton solution which has loop-like form, i. e., it is a multiple-valued function (see Fig. 1 in [1]). Recently, we obtained the two loop soliton solution to the VE both by use of Hirota's method [4] and by use of elements of the inverse scattering transform (IST) procedure for the KdV equation [5, 6]. We have obtained the N loop soliton solution to the VE by use of Hirota's method [MoPa2]. Whilst these multi-soliton solutions are rather intriguing, it is the solution to the initial value problem that is of more interest in a physical context. As we have shown that the VE is integrable, the IST is the most appropriate way of tackling the initial value problem. In order to use the IST method, one first has to formulate the associated eigenvalue problem; this is the main aim of this paper. We achieve this aim by first finding a Bäcklund transformation associated with the VE. It is well known that the Bäcklund transformation is one of the analytic tools for dealing with soliton problems and has a close relationship to the IST method [8, 9].

In §2, we introduce new independent coordinates as previously [4, 5]. In terms of these coordinates, the solution to the VE is given by single-valued parametric relations. The transformation into these coordinates is the key to solving the problem of the interaction of solitons as well as explaining multiple-valued solutions [2]. This transformation also leads to an equation that can be expressed in bilinear form in terms of the Hirota D operator [8]. In §3, we present the Bäcklund transformation both in bilinear form and in ordinary form for the VE written in terms of the new independent variables. This type of Bäcklund transformation was first introduced by Hirota [9] and has the advantage that the transformation equations are linear with respect to each dependent variable. The Bäcklund transformation is rewritten in ordinary form which enables one to relate pairs of solutions of the VE. In §4, we find that the IST problem for the transformed VE involves a third order eigenvalue problem. As an example, we state the one soliton solution of the VE as found by using the IST method.

2. Equation in new coordinates. As previously [4, 5], let us define new independent variables (T, X) by the transformations

$$\varphi\, dT = dx - u\, dt, \qquad X = t. \tag{2}$$

The function φ is to be obtained. It is an additional dependent variable in the system of equations (4), (5) to which we reduce the original equation (1). Transformation (2) is similar to the transformation between Eulerian coordinates (x, t) and Lagrangian coordinates (T, X). We then require $T = x$ if there is no perturbation, i. e., if $u(x, t) = 0$. Hence, $\varphi = 1$ when $u(x, t) = 0$. For example, it may be shown from Eqs. (12) and (14) in [1] that $\varphi = 1 - u/v$ for the one loop soliton solution. It is noted that the functions $x = \theta(T, X)$ and $u = U(T, X)$ are single-valued.

In terms of the coordinates (T, X), Eq. (1) in the unknown $U(T, X) \equiv u(x, t)$ has the form

$$\varphi^{-1} \frac{\partial}{\partial T} \frac{\partial}{\partial X} U + U = 0. \tag{3}$$

The equation for the variable φ can be obtained in the following way. Noting that the transformation inverse to (2) is

$$dx = \varphi\, dT + U\, dX,$$

and taking into account the condition that dx is an exact differential, we get

$$\frac{\partial \varphi}{\partial X} = \frac{\partial U}{\partial T}. \tag{4}$$

This equation together with Eq. (3), rewritten in the form

$$\frac{\partial^2 \varphi}{\partial X^2} + U\varphi = 0, \tag{5}$$

are the main system of equations. In terms of the coordinates (T, X), the solution is given by single-valued parametric relations. The transformation into these coordinates is the key to solving the problem of the interaction of solitons as well as explaining multiple-valued solutions [2]. System (4), (5) can be reduced to a nonlinear equation in the unknown W defined by

$$W_X = U \tag{6}$$

as follows. As in [4, 5], we study solutions U that vanish as $|X| \to \infty$ or, equivalently, solutions for which W tends to a constant as $|X| \to \infty$. From (4) and (6) and the requirement that $\varphi \to 1$ as $|X| \to \infty$, we have $\varphi = 1 + W_T$; then (5) may be written (the transformed Vakhnenko equation) as

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{7}$$

Furthermore, then it follows that the original independent space coordinate x is given by

$$x = \theta(T, X) := x_0 + T + W, \tag{8}$$

where x_0 is an arbitrary constant.

Finally, by taking

$$W = 6(\ln f)_X, \tag{9}$$

where f is a function of X and T, we observe that the transformed VE (7) may be written as the bilinear equation [4]

$$(D_T D_X^3 + D_X^2) f \cdot f = 0. \tag{10}$$

3. Bäcklund transformation for the transformed Vakhnenko equation. In this section, we present a Bäcklund transformation for Eq. (10), the bilinear form of the transformed VE (7).

We follow the method developed in [9]. First, we define P as follows:

$$P := 2\left\{\left[(D_T D_X^3 + D_X^2) f' \cdot f'\right] ff - f'f'\left[(D_T D_X^3 + D_X^2) f \cdot f\right]\right\}, \tag{11}$$

where $f \neq f'$. We aim to find a pair of equations such that each equation is linear in each of the dependent variables f and f', and such that together f and f' satisfy $P = 0$. (It follows then that if f is a solution of (10), then so is f' and vice versa.) The pair of equations is the required Bäcklund transformation.

We show that the Bäcklund transformation is given by two equations

$$(D_X^3 - \lambda) f' \cdot f = 0, \tag{12}$$

$$(3D_X D_T + 1 + \mu D_X) f' \cdot f = 0, \tag{13}$$

where $\lambda = \lambda(X)$ is an arbitrary function of X and $\mu = \mu(T)$ is an arbitrary function of T.

We prove that together f and f', as determined by Eqs. (12), (13), satisfy $P = 0$ as follows. By using identities (VII.3), (VII.4) from [10], and Eq. (5.86) from [8], we may express P in the following form:

$$P = D_T\left[(D_X^3 f' \cdot f) \cdot (f'f) - 3(D_X^2 f' \cdot f) \cdot (D_X f' \cdot f)\right] + D_X\left[3(D_T D_X^2 f' \cdot f) \cdot (f'f) - \right.$$
$$\left. - 6(D_X D_T f' \cdot f) \cdot (D_X f' \cdot f) - 3(D_X^2 f' \cdot f) \cdot (D_T f' \cdot f) + 4(D_X f' \cdot f) \cdot (f'f)\right]. \tag{14}$$

By using (26) and (27), we can rewrite P in the following form, where $\lambda = \lambda(X)$ and $\mu = \mu(T)$:

$$P = 4D_T\left(\left\{D_X^3 - \lambda(X)\right\} f' \cdot f\right) \cdot (f'f) -$$
$$- 4D_X\left(\left\{3D_T D_X + 1 + \mu(T) D_X\right\} f' \cdot f\right) \cdot (D_X f' \cdot f). \tag{15}$$

In constructing this expression for P, we have used the results

$$D_X a \cdot b = -D_X b \cdot a \qquad \text{and} \qquad D_X a \cdot a = 0,$$

which follow from result (II) in [8]. It is clear from (15) that if Eqs. (12), (13) hold then $P = 0$ as required.

Thus, we have proved that Eqs. (12), (13) constitute the Bäcklund transformation for Eq. (10). Separately these equations appear as a part of the Bäcklund transformation for other nonlinear evolution equations. For example, Eq. (12) is the same as one of the equations that is a part of the Bäcklund transformation for a higher order KdV equation (see Eq. (5.139) in [8]), and Eq. (13) is similar to (5.132) in [8] that is a part of the Bäcklund transformation for a model equation for shallow water waves.

The inclusion of μ in the operator $3D_T + \mu$ which appears in (15) corresponds to a multiplication of f and f' by terms of the form $e^{g(T)}$ and $e^{g'(T)}$, respectively; we see from (9) that this has no effect on W or W'. Hence, without loss of generality, we may take $\mu = 0$ in Eq. (13) if we wish.

Following the procedure given in [8, 11], we can rewrite the Bäcklund transformation in ordinary form in terms of the potential $W = \int\limits_{-\infty}^{X} U\,dX'$ obtained from (6). In new variables defined by

$$\phi = \ln f'/f, \qquad \rho = \ln f'f, \tag{16}$$

Eqs. (12), (13) have the form

$$\phi_{XXX} + 3\phi_X\rho_{XX} + \phi_X^3 - \lambda = 0, \tag{17}$$

$$3(\rho_{XT} + \phi_X\phi_T) + 1 + \mu\phi_X = 0 \tag{18}$$

respectively, where we have used results similar to (XI.1) – (XI.3) in [8]. From definitions (9) and (16), different solutions W, W' of Eq. (7) are related to ϕ and ρ by

$$W' - W = 6\phi_X, \qquad W' + W = 6\rho_X. \tag{19}$$

Substitution of (19) into (17), (18) with $\mu = 0$ leads to

$$(W' - W)_{XX} + \frac{1}{2}(W' - W)(W' + W)_X + \frac{1}{36}(W' - W)^3 - 6\lambda = 0, \tag{20}$$

$$(W' - W)\left[3(W' + W)_{XT} + \frac{1}{2}(W' - W)(W' - W)_T\right] - \\ - 6(W' - W)_X\left[1 + \frac{1}{2}(W' + W)_T\right] = 0. \tag{21}$$

The required Bäcklund transformation is Eqs. (20), (21).

4. Formulation of the inverse scattering eigenvalue problem. In this section, we will show that the IST problem for the transformed VE in the form (7) has a third order eigenvalue problem that is similar to the one associated with a higher order KdV equation [11, 12], a Boussinesq equation [12, 13, 15], and a model equation for shallow water waves [8, 10].

Introducing the function

$$\psi = f'/f, \tag{22}$$

and taking into account Eqs. (6) and (9), we find that Eqs. (12), (13) reduce to

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{23}$$

$$3\psi_{XT} + (W_T + 1)\psi + \mu\psi_X = 0 \tag{24}$$

respectively, where we have used results similar to (X.1)–(X.3) in [8]. It may be shown from (23) and (24) that, even with $\mu \neq 0$,

$$[W_{XXT} + (1 + W_T)W_X]_X\psi + 3\lambda_X\psi_T = 0.$$

Hence, (7) is the condition for $\lambda_X = 0$ and for λ to be constant. The constant λ is what is required in the IST problem. Since Eqs. (23), (24) are alternative forms of Eqs. (12), (13), it follows that system (23), (24) is associated with the transformed VE (7) considered here. Thus, the IST problem is directly related to a spectral equation of third order, namely (23). The inverse problem for certain third order spectral equations has been considered by Kaup [12] and Caudrey [13, 14].

As an example, we state the one soliton solution for the transformed VE that follows from Eqs. (3.29) and (3.30) in [12]. In our case, the time dependence of ψ is described by Eq. (24) in contrast to Eq. (1.2) from [12] for the 5th order KdV equation. Taking these features into account, we can rewrite Eq. (3.30) from [12] for the transformed VE with the notation used in [4] as

$$U = 6Q = 6 \cdot \frac{3}{4}\eta^2 \operatorname{sech}^2\left[\frac{\sqrt{3}}{2}\eta\left(X - \frac{T}{3\eta^2}\right)\right] = 6k^2 \operatorname{sech}^2\left[k\left(X - \frac{T}{4k^2}\right)\right]. \tag{25}$$

(25) is essentially the one soliton solution of the transformed VE (7) that is given in [4] by Eq. (3.4).

5. Conclusion. We have found the Bäcklund transformation both in bilinear form and in ordinary form for the transformed VE. It enables us to formulate an IST problem for the transformed VE which is directly related to a spectral equation of third order. We have stated the one soliton solution of the transformed VE as derived by use of the IST method. The corresponding result for the N soliton solution is currently under investigation, as is the general initial value problem.

6. Appendix. The following identities (26), (27) are required in §3:

$$D_X^3(D_Tf'\cdot f)\cdot(ff') = D_T\left[(D_X^3 f'\cdot f)\cdot(ff') - 3(D_X^2 f'\cdot f)\cdot(D_X f'\cdot f)\right]. \tag{26}$$

$$\begin{aligned} 4D_T(D_X^2 f'\cdot f)\cdot(D_X f'\cdot f) &= D_X[(D_TD_X^2 f'\cdot f)\cdot(f'f) + 2(D_TD_X f'\cdot f)\cdot(D_X f'\cdot f) - \\ &- (D_X^2 f'\cdot f)\cdot(D_T f'\cdot f)] - D_X^3(D_T f'\cdot f)\cdot(f'f). \end{aligned} \tag{27}$$

Identities (26) and (27) come from

$$\begin{aligned} \exp(D_1)[\exp(D_2)f'\cdot f]\cdot[\exp(D_3)f'\cdot f] &= \exp\left(\frac{1}{2}\{D_2 - D_3\}\right) \times \\ &\times \left[\exp\left\{\frac{1}{2}(D_2 + D_3) + D_1\right\} f'\cdot f\right]\cdot\left[\exp\left\{\frac{1}{2}(D_2 + D_3) - D_1\right\} f'\cdot f\right], \end{aligned} \tag{28}$$

which is Eq. (5.83) in [8], where $D_i = \varepsilon_i D_X + \delta_i D_T$. In the order $\varepsilon_1^3\delta_3$, (28) yields (26), and (28) yields (27) in the order $\delta_1\varepsilon_2^2\varepsilon_3$.

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