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A novel nonlinear evolution equation integrable by the inverse scattering method

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В роботі для еволюційного рівняння $(u_t + uu_x)_x + u = 0$ формулюється обернена задача розсіювання. Асоційована система рівнянь утримує спектральне рівняння третього порядку. Методом оберненої задачі розсіювання знайдено точний N-солітонний розв'язок рівняння Вахненка.

In this paper, we continue to study the nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \tag{1}$$

This equation was first suggested by Vakhnenko in [1] to describe high-frequency waves in a relaxing medium [2]. Hereafter, as was initiated in [3], Eq. (1) is referred to as the Vakhnenko equation (VE). A number of papers deal with the VE. In [1, 3], travelling-wave solutions were investigated, while the symmetry properties of the equation were studied in [4]. The physical interpretation of multivalued functions that describe loop-like soliton solutions was given in [2]. The loop-like solutions are stable to long-wavelength perturbations [3]. The introduction of a dissipative term, with a dissipation parameter less than some limit value, does not destroy these loop-like solutions [2]. Two-loop soliton solutions have been obtained both by use of Hirota's method [5] and by use of elements of the inverse scattering transform (IST) procedure for the KdV equation [6]. We have also applied Hirota's method to obtain the N-soliton solution and to prove that the VE is integrable [7]. As the IST method is the most appropriate way of tackling the initial-value problem, we have formulated the associated eigenvalue problem for the transformed VE [8]. This was achieved by finding a Bäcklund transformation associated with the VE. It turns out that the IST problem is directly related to a spectral equation of third order. The inverse problem for certain third-order spectral equations was considered by Kaup [9] and Caudrey [10, 11]. These results enable us in this paper to find N-soliton solutions to the transformed VE as derived by using the IST method.

It is convenient to write Eq. (1) in new independent coordinates X, T as defined in [5], namely

$$x = x_0 + T + W(X,T), \qquad t = X, \qquad W = \int_{-\infty}^{X} U(X',T)\,dX'. \tag{2}$$

Here, $u(x,t) = U(X,T)$, and x_0 is a constant. We also assume that, as $X \to -\infty$, the derivatives of W vanish and W tends to a constant. Eq. (1) then has the form [5, 8]

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{3}$$

If the solution $U(X,T) = W_X$ of the transformed VE (3) is obtained, the original independent space coordinate x can be found by means of formula (2). This relationship together with

$u(x,t) = U(X,T)$ enables us to define a solution of the VE (1) in parametric form with T as a parameter. We note that transformation (2) between the old and new coordinates is similar to the transformation between the Eulerian variables (x,t) and Lagrangian variables (T,X) [8].

It is well known that the Bäcklund transformation is one of the analytic tools for dealing with soliton problems and has a close relationship to the IST method [12, 13]. In [8], we have formulated the IST problem for the transformed VE in the form (3) as

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{4}$$

$$3\psi_{XT} + (1 + W_T)\psi = 0, \tag{5}$$

where λ is a constant. We achieved this by finding a Bäcklund transformation associated with Eq. (3) [8]. The condition for the compatibility of Eqs. (4) and (5) is Eq. (3). Thus, the IST problem is directly related to a spectral equation of third order (4). The third-order eigenvalue problem is similar to the one associated with a higher order KdV equation [9, 14], a Boussinesq equation [9, 10], and a model equation for shallow water waves [12, 15]. Kaup [9] and Caudrey [10, 11] studied the inverse problem for certain third-order spectral equations.

The general theory of the inverse scattering problem for N spectral equations was developed in [10]. Following that paper, the spectral equation (4) can be rewritten as

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X,\zeta)]\cdot\boldsymbol{\psi} \tag{6}$$

by putting

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -U & 0 \end{pmatrix}. \tag{7}$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_j(\zeta)$ and left- and right-eigenvectors $\widetilde{\mathbf{v}}_j(\zeta)$ and $\mathbf{v}_j(\zeta)$, respectively, where

$$\lambda_j(\zeta) = \omega_j\zeta, \qquad \lambda_j^3(\zeta) = \lambda, \qquad \mathbf{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \end{pmatrix}, \qquad \widetilde{\mathbf{v}}_j(\zeta) = \left(\lambda_j^2,\ \lambda_j,\ 1\right), \tag{8}$$

and $\omega_j = e^{i\frac{2\pi}{3}(j-1)}$ are the cubic roots of 1.

A solution of the linear Eq. (3) (or, equivalently, Eq. (6)) was obtained by Caudrey [10] in terms of Jost functions $\boldsymbol{\phi}_j(X,\zeta)$ which have the asymptotic behaviour

$$\boldsymbol{\Phi}_j(X,\zeta) := \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X,\zeta) \to \mathbf{v}_j(\zeta) \qquad \text{as} \quad X \to -\infty. \tag{9}$$

Here, T is regarded as a parameter until the T-evolution of scattering data is taken into account later. The solution of the direct problem is given by the system of equations (4.5) in [10]. We restrict our attention to the N-soliton solution. To do this, we consider Eq. (6.20) from [10] by putting $Q_{ij}(\zeta) \equiv 0$. Then there is only the bound-state spectrum which is associated with soliton solutions.

Let the bound-state spectrum be defined by K poles. Relation (4.5) from [10] is reduced to

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)}\Phi_1(X,\omega_j\zeta_1^{(k)}). \tag{10}$$

We need only to consider the function $\Phi_1(X,\zeta)$ since there is a set of symmetry properties as those for the Boussinesq equation, namely, properties (6.15) in [10] for Jost functions $\phi_j(X,\zeta)$. Eqs. (10) involve the spectral data: poles $\zeta_i^{(k)}$ and quantities $\gamma_{ij}^{(k)}$. First, we prove that $\operatorname{Re}\lambda = 0$ for a compact support. Indeed, from Eq. (4), we have

$$(\psi_X)_{XXX} + (U\psi_X)_X - \lambda\psi_X = 0. \tag{11}$$

This together with Eq. (4) enables us to write

$$\frac{\partial}{\partial X}\left(\frac{\partial^2}{\partial X^2}\psi_X\psi^* - 3\psi_{XX}\psi_X^* + U\psi_X\psi^*\right) - 2\operatorname{Re}\lambda\psi_X\psi^* = 0. \tag{12}$$

Integrating Eq. (12) over all values of X, we obtain that, for a compact support, Eq. (4) has only the bound-state spectrum, and $\operatorname{Re}\lambda = 0$ since, in the general case, $\int\limits_{-\infty}^{\infty}\psi_X\psi^*\,dX \neq 0$.

As follows from Eqs. (2.12), (2.13), (2.36), and (2.37) of [9], $\psi_X(\zeta)$ is related to the adjoint states $\psi^A(-\zeta)$. In the usual manner, using the adjoint states and Eq. (14) from [11] and Eq. (2.37) from [9], one can obtain

$$\phi_{1X}(X,\zeta) = \frac{\mathrm{i}}{\sqrt{3}}\left[\phi_{1X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) - \phi_{1X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)\right]. \tag{13}$$

It is easily seen that if $\zeta_1^{(1)}$ is a pole of $\phi_1(X,\zeta)$, then there is a pole either at $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_2\zeta)$ has a pole) or at $\zeta_1^{(2)} = -\omega_3\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_3\zeta)$ has a pole). For definiteness, let $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$, then Eq. (13) implies that the point $-\omega_3\zeta_1^{(2)}$ should be a pole. However, this pole coincides with the pole $\zeta_1^{(1)}$ since $-\omega_3\zeta_1^{(2)} = -\omega_3(-\omega_2)\zeta_1^{(1)} = \zeta_1^{(1)}$. Hence, poles appear in the pairs $\zeta_1^{(2n-1)}$, $\zeta_1^{(2n)}$ under the condition $\zeta_1^{(2n)}/\zeta_1^{(2n-1)} = -\omega_2$, where n is the number pair.

Let us consider N pairs of poles, i.e., there are $K = 2N$ poles in all over which the sum is taken in Eqs. (10). For the pair n $(n = 1,2,\ldots,N)$, we have

$$\text{(i)}\quad \zeta_1^{(2n-1)} = \mathrm{i}\omega_2\xi_n, \qquad \text{(ii)}\quad \zeta_1^{(2n)} = -\mathrm{i}\omega_3\xi_n. \tag{14}$$

Since U is real and λ is imaginary, ξ_k is real. Relationships (14) are in line with condition (2.33) from [9]. These relationships are also similar to Eqs. (6.24) and (6.25) in [10], while $\gamma_{1j}^{(k)}$ turns out to be different from $\tilde{\gamma}_{1j}^{(k)}$ for the Boussinesq equation (see Eqs. (6.24) and (6.25) in [10]). Indeed, by considering Eq. (13) in the vicinity of the first pole $\zeta_1^{(2n-1)}$ of the pair n and using relation (10), one can obtain a relation between $\gamma_{12}^{(2n-1)}$ and $\gamma_{13}^{(2n)}$. In this case, the functions $\phi_{1X}(X,\zeta)$, $\phi_1(X,-\omega_2\zeta)$, and $\phi_{1X}(X,-\omega_2\zeta)$ also have poles here, while the functions $\phi_1(X,-\omega_3\zeta)$ and $\phi_{1X}(X,-\omega_3\zeta)$ have no ones. Substituting $\phi_1(X,\zeta)$ in the form (9), (10) in Eq. (13) and letting $X \to -\infty$, we have the ratio $\gamma_{13}^{(2n)}/\gamma_{12}^{(2n-1)} = \omega_2$ and $\gamma_{12}^{(2n)} = \gamma_{13}^{(2n-1)} = 0$. Therefore, the properties of $\gamma_{ij}^{(k)}$ should be defined by the relationships

$$\left.\begin{aligned} &\text{(i)}\quad \gamma_{12}^{(2n-1)} = \omega_2\beta_k, \qquad \gamma_{13}^{(2n-1)} = 0,\\ &\text{(ii)}\quad \gamma_{12}^{(2n)} = 0, \qquad \gamma_{13}^{(2n)} = \omega_3\beta_k, \end{aligned}\right\} \tag{15}$$

where, as will be proved below, β_k is real when U is real.

By expanding $\Phi_1(X,\zeta)$ as an asymptotic series in $\lambda_1^{-1}(\zeta)$, one can obtain (cf. Eq. (2.7) in [9])

$$\Phi_1(X,\zeta) = 1 - \frac{1}{3\lambda_1(\zeta)}\left[W(X) - W(-\infty)\right] + O(\lambda_1^{-2}(\zeta)). \tag{16}$$

On the other hand, we may rewrite relationship (10) as (see, for instance, Eqs. (6.33) and (6.34) in [10])

$$\begin{aligned}\Phi_1(X,\zeta) &= 1 - \sum_{k=1}^{2N} \frac{\exp\{-\lambda_1(\zeta_1^{(k)})X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)}\Psi_k(X),\\ \Psi_k(X) &= \sum_{j=2}^{3}\gamma_{1j}^{(k)}\exp\{\lambda_j(\zeta_1^{(k)})X\}\Phi_1(X,\omega_j\zeta_1^{(k)}),\end{aligned} \tag{17}$$

It follows from (16) and (17) that (cf. Eq. (6.38) in [10])

$$W(X) - W(-\infty) = -3\sum_{k=1}^{2N}\exp\{-\lambda_1(\zeta_1^{(k)})X\}\Psi_k(X) = 3\frac{\partial}{\partial X}\ln(\det M). \tag{18}$$

The matrix M is defined as that in relationship (6.36) in [10] by

$$M_{kl}(X) = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})]X\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})}. \tag{19}$$

Now let us consider the T-evolution of spectral data. By analyzing the solution of Eq. (5) when $X \to -\infty$, we find that $\phi_i(X,T,\zeta) = \exp\left[-(3\lambda_i(\zeta))^{-1}T\right]\phi_i(X,0,\zeta)$. Hence, the T-evolution of scattering data is given by the relationships ($k = 1,2,\ldots,K$)

$$\left.\begin{aligned}&\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0),\\ &\gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0)\exp\{[-(3\lambda_j(\zeta_1^{(k)}))^{-1} + (3\lambda_1(\zeta_1^{(k)}))^{-1}]T\}.\end{aligned}\right\} \tag{20}$$

The final result, including the T-evolution, for the N-soliton solution of the transformed VE is

$$U(X,T) = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right), \tag{21}$$

where M is a $2N \times 2N$ matrix given by

$$\begin{aligned}M_{kl} &= \delta_{kl} - \\ &-\sum_{j=2}^{3}\gamma_{1j}^{(k)}(0)\frac{\exp\{[-(3\lambda_j(\zeta_1^{(k)}))^{-1} + (3\lambda_1(\zeta_1^{(k)}))^{-1}]T + (\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)}))X\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})},\end{aligned} \tag{22}$$

and

$$n = 1,2,\ldots,N, \qquad m = 2n-1,$$

$$\lambda_1(\zeta_1^{(m)}) = \mathrm{i}\omega_2\xi_m, \qquad \lambda_2(\zeta_1^{(m)}) = \mathrm{i}\omega_3\xi_m, \qquad \gamma_{12}^{(m)}(0) = \omega_2\beta_m, \qquad \gamma_{13}^{(m)}(0) = 0,$$
$$\lambda_1(\zeta_1^{(m+1)}) = -\mathrm{i}\omega_3\xi_m, \quad \lambda_3(\zeta_1^{(m+1)}) = -\mathrm{i}\omega_2\xi_m, \quad \gamma_{12}^{(m+1)}(0) = 0, \quad \gamma_{13}^{(m+1)}(0) = \omega_3\beta_m.$$

For the N-soliton solution, there are N arbitrary constants ξ_m and N arbitrary constants β_m. Note that $m = 1, 3, \ldots, 2N-1$, i.e., m is odd.

In order to obtain a one-soliton solution of the transformed VE (3), we need first to calculate the 2×2 matrix M according to Eq. (22) with $N = 1$. We find that the matrix is

$$\begin{pmatrix} 1 - \dfrac{\omega_2\beta_1}{\sqrt{3}\xi_1}\exp[\sqrt{3}\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] & \dfrac{\mathrm{i}\omega_3\beta_1}{2\xi_1}\exp[2\mathrm{i}\omega_3\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] \\ \dfrac{-\mathrm{i}\omega_2\beta_1}{2\xi_1}\exp[-2\mathrm{i}\omega_2\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] & 1 - \dfrac{\omega_3\beta_1}{\sqrt{3}\xi_1}\exp[\sqrt{3}\xi_1 X - (\sqrt{3}\xi_1)^{-1}T] \end{pmatrix}, \tag{23}$$

and its determinant is

$$\det M = \left\{1 + \frac{\beta_1}{2\sqrt{3}\xi_1}\exp\left[\sqrt{3}\xi_1\left(X - \frac{T}{3\xi_1^2}\right)\right]\right\}^2. \tag{24}$$

Consequently, from Eq. (21), the one-soliton solution of the transformed VE as obtained by the IST method is

$$U = \frac{9}{2}\xi_1^2\,\mathrm{sech}^2\left[\frac{\sqrt{3}}{2}\xi_1\left(X - \frac{T}{3\xi_1^2}\right) + \alpha_1\right], \tag{25}$$

where $\alpha_1 = \frac{1}{2}\ln(\beta_1/2\sqrt{3}\xi_1)$ is an arbitrary constant. Since U is real, it follows from (25) that α_1 is real, and so β_1 is also real. Recently, we found the same solution by means of Hirota's method (see Eq. (3.4) in [5]).

It is of interest to compare Eq. (25) with the solution of the 5th-order KdV-like equation discussed in [9]. The spectral equation (4) is the same as that given by (1.1) (with $R = 0$) in [9], whereas the equation that governs the time-dependence of ψ, i.e. (5), is different from (1.2) in [9]. Thus, the X-dependence of (25) should agree with the x-dependence of the solution given by (3.30) in [9]. With the identification $U = 6Q$, $\xi_1 = \eta$, this is indeed the case.

Let us now consider a two-soliton solution of the transformed VE. In this case, M is a 4×4 matrix. We give no its explicit form here, but we find that

$$\det M = \left(1 + q_1^2 + q_2^2 + b^2 q_1^2 q_2^2\right)^2, \tag{26}$$

where

$$q_i = \exp\left[\frac{\sqrt{3}}{2}\xi_i\left(X - \frac{T}{3\xi_i^2}\right) + \alpha_i\right], \qquad b^2 = \left(\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1}\right)^2 \frac{\xi_1^2 + \xi_2^2 - \xi_1\xi_2}{\xi_1^2 + \xi_2^2 + \xi_1\xi_2}, \tag{27}$$

and $\alpha_i = \frac{1}{2}\ln(\beta_i/2\sqrt{3}\xi_i)$ are arbitrary constants. Both for the one-soliton solution (24) and for the two-soliton solution (26), $\det M$ is a perfect square. The two-soliton solution to the transformed VE is given by (21) together with (26). In contrast to the IST method that we have used here, Hirota's method was used in [5] to obtain the same two-soliton solution.

This paper completes the series of papers that connect the application of the IST method to the VE [6, 8]. The study of the evolution and formation of multivalued solutions (loop-like solutions) from a single-valued initial solution is important and will be investigated in the future.

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