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# The solutions of a generalized Degasperis–Procesi equation

*(Presented by Academician of the NAS of Ukraine A. G. Zagorodny)*

*У різних фізичних розділах з'являються рівняння Камасса–Холма (CH рівняння) та рівняння Дегаспереса–Процесі (DP рівняння). Ці рівняння, як відомо, використовуються для опису хвиль на мілкій воді, турбулентних потоків, а також хвильових процесів у релаксуючому середовищі. Запропоновано нове нелінійне еволюційне рівняння, яке узагальнює DP і CH рівняння. Це рівняння вдається інтегрувати подібно до DP і CH рівнянь, щоб одержати розв'язки на біжучих хвилях. Класифікація розв'язків, яка проведена в роботі, може бути корисною для розуміння і опису фізичних процесів, що досліджуються. Показано, що розв'язки нового рівняння можуть бути інтерпретовані, як проекції спіралі на площину під різними кутами до осі спіралі. Розв'язки у вигляді відокремлених хвиль з'являються, коли розглядається одна петля спіралі з розтягнутими верхньою або/та нижньою частинами.*

**1. Introduction.** We proceed from the family of the equations

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \qquad (1.1)$$

known as the 'peakon $b$-family' [1]. As is proved in [2], only two equations from family (1.1) are integrable, namely the ones for which $b = 2$ and $b = 3$. With $b = 2$, Eq. (1.1) is known as the Camassa–Holm equation (CHe) [3]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad (1.2)$$

and, with $b = 3$, it is known as the Degasperis–Procesi equation (DPe) [4]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \qquad (1.3)$$

Originally Eq. (1.2) was derived as an equation for shallow water waves [3]. Later Chen et al. showed that Eq. (1.2) can be applied successfully to describe turbulent flows [5]. Since Hone and Wang revealed the connection of the DPe and the Vakhnenko equation [6, 7], Eq. (1.3) can be used to model wave perturbations in relaxing media. As proved by Lenells (and it is important from the physical point of view), the multivalued solutions of the CHe and the DPe can be the basis for the construction of one-valued solutions [8].

It turns out that the travelling-wave solutions for both Eq. (1.2) and Eq. (1.3) can be written in terms of the same formulas (see (2.5), (2.6) in Section 2) [9, 10]. In Section 3, we suggest a new nonlinear equation that can be integrated in a similar way. There we present a classification of the travelling-wave solutions of this new equation. Finally, we indicate in Section 4 that the solutions can be interpreted as the projection of a spiral on a plane at different projection angles to the axis of the spiral. The solitary-wave solutions appear when we consider a single loop of a spiral with extended upper or/and lower parts.

**2. Travelling-wave solutions of CHe and DPe.** Recently, we studied Eqs. (1.2) and (1.3) and obtained the travelling-wave solutions [9, 10]. We repeat briefly some results from these papers for convenience.

Restricting our attention to travelling waves, we introduce new variables

$$z = \frac{u - v}{|v|}, \qquad \eta = x - vt - x_0, \tag{2.1}$$

where $v$ and $x_0$ are arbitrary constants, and $v \neq 0$. In these variables, Eq. (1.1) has the form

$$zz_{\eta\eta\eta} + bz_\eta z_{\eta\eta} - (b+1)zz_\eta - bcz_\eta = 0, \qquad \text{with} \qquad c = \pm 1, \tag{2.2}$$

and can be integrated twice to give

$$(zz_\eta)^2 = f(z), \qquad \text{with} \qquad f(z) = z^4 + 2cz^3 + Az^2 + Bz^{3-b}.$$

For CHe (1.2), we have

$$f(z) = z^4 + 2cz^3 + Az^2 + Bz = (z - z_1)(z - z_2)(z - z_3)(z - z_4), \tag{2.3}$$

while, for DPe (1.3), we obtain

$$f(z) = z^4 + 2cz^3 + Az^2 + B = (z - z_1)(z - z_2)(z - z_3)(z - z_4). \tag{2.4}$$

Here $f(z)$ are polynomials of the fourth order, and the $z_i$ are the roots; in general, they are different for (2.3) and (2.4).

Following [9, 10], we write here two forms of the solutions in terms of the roots of the polynomial. These solutions are appropriate when the $z_i$ are real and $z$ is such that $z_1 \leqslant z_2 \leqslant$ $\leqslant z \leqslant z_3 \leqslant z_4$. The solutions are in parametric form with $w$ as a parameter.

The first form of the solution is as follows:

$$z = \frac{z_2 - z_1 n \operatorname{sn}^2(w|m)}{1 - n \operatorname{sn}^2(w|m)}, \tag{2.5}$$

with

$$n = \frac{z_3 - z_2}{z_3 - z_1}, \qquad p = \frac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)}, \qquad m = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)},$$

and

$$\eta = \frac{wz_1 + (z_2 - z_1)\Pi(n; w|m)}{p}.$$

Here $\operatorname{sn}(w|m)$ is a Jacobian elliptic function, and $\Pi(n; w|m)$ is an elliptic integral of the third kind.

The second form of the solution is

$$z = \frac{z_3 - z_4 n \operatorname{sn}^2(w|m)}{1 - n \operatorname{sn}^2(w|m)}, \tag{2.6}$$

with

$$n = \frac{z_3 - z_2}{z_4 - z_2}, \qquad \eta = \frac{wz_4 - (z_4 - z_3)\Pi(n; w|m)}{p},$$

where $p$ and $m$ are as in (2.5).

**3. The generalized Degasperis–Procesi equation.** Since the different polynomials in (2.3) and (2.4) can be written in the same form, as is shown by the right-hand sides in (2.3) and (2.4), we anticipate that there is a nonlinear equation which, for travelling-wave solutions, will reduce to

$$(zz_\eta)^2 = f(z), \tag{3.1}$$

with

$$f(z) = z^4 + 2cz^3 + Az^2 + Dz + B = (z - z_1)(z - z_2)(z - z_3)(z - z_4).$$

It follows that this equation should be solvable in a way similar to that for the CHe and the DPe.

Let us consider the new nonlinear evolution equation

$$(u_t + uu_x)^{b-1}(u_t - u_{txx} + (b+1)uu_x - bu_xu_{xx} - uu_{xxx}) + \frac{1}{2}(2-b)D|v|^b u_x^b = 0. \tag{3.2}$$

Equation (3.2) generalizes Eq. (1.1) due to the inclusion of an additional factor and an additional term. For travelling waves, Eq. (3.2) in terms of variables (2.1) has the form

$$z^{b-1}(zz_{\eta\eta\eta} + bz_\eta z_{\eta\eta} - (b+1)zz_\eta - bcz_\eta) - \frac{1}{2}(2-b)Dz_\eta = 0, \qquad \text{with} \qquad c = \pm 1. \tag{3.3}$$

It can be seen that Eq. (3.3) generalizes Eq. (2.2) due to the inclusion of an additional factor $z^{b-1}$ and an additional term $\frac{1}{2}(2-b)Dz_\eta$. After two integrations, we get

$$(zz_\eta)^2 = f(z), \qquad \text{with} \qquad f(z) = z^4 + 2cz^3 + Az^2 + Dz^{4-b} + Bz^{3-b}. \tag{3.4}$$

With $b = 3$, Eq. (3.2) becomes

$$(u_t + uu_x)^2(u_t - u_{txx} + 4uu_x - 3u_xu_{xx} - uu_{xxx}) - \frac{1}{2}D|v|^3u_x^3 = 0. \tag{3.5}$$

This will be referred hereafter as the generalized Degasperis–Procesi equation (gDPe). Also, with $b = 3$, Eq. (3.3) becomes

$$z^2(zz_{\eta\eta\eta} + 3z_\eta z_{\eta\eta} - 4zz_\eta - 3cz_\eta) + \frac{1}{2}Dz_\eta = 0$$

and Eq. (3.4) becomes Eq. (3.1). Eq. (3.1) with $B = 0$ corresponds to the CHe, for which $f(z)$ is given by (2.3); Eq. (3.1) with $D = 0$ corresponds to the DPe, for which $f(z)$ is given by (2.4).

In principle, as the polynomial in (3.1) is a quartic, we can use the method of integration we applied to the CHe and the DPe to integrate gDPe (3.5) and obtain travelling-wave solutions in the forms given by Eqs. (2.5) or (2.6).

It is necessary to note that $f(z)$ in (3.1) involves three arbitrary constants $A$, $B$, $D$ in contrast to $f(z)$ in (2.3) and (2.4), where there are only two constants. Hence, the gDPe should possess a wider variety of travelling-wave solutions than either the CHe or the DPe.

Since Eq. (3.1) is invariant under the transformation $z \to -z$, $c \to -c$, $D \to -D$, we can consider only the case $c = 1$ (i. e. $v > 0$). Note that there is a restriction on the roots; they cannot be arbitrary because $z_1 + z_2 + z_3 + z_4 = -2c$, and they must be real.

In Table 1, we classify the different types of travelling-wave solutions of gDPe (3.5) according to the disposition of the real roots of the polynomial $f(z)$. With distinct roots, the solutions are shown in the first column (Figs. 1.1–1.5). When $z_1 \neq z_2$ and $z_3 = z_4$, the solutions take the forms which are shown in the second column (Figs. 2.1–2.5). When $z_1 = z_2$ and $z_3 \neq z_4$, the solutions are shown in the third column (Figs. 3.1–3.3). Finally, in the fourth column, there are the solutions with $z_1 = z_2$ and $z_3 = z_4$ (Figs. 4.1–4.3).

It should be noted that it is possible to construct other explicit solutions as composite waves by using separate parts of the solutions from Table 1 [8]. Examples of this procedure have been given in Appendix B in [10]. In particular, the circle in Fig. 1.3b can be used to construct a periodic bell-like solution (see Fig. 3 (c) in [9]), while the two-valued solution in Figs. 4.3 a, 4.3 b can be used to construct a kink-like solution with infinite slope (see Fig. 4(c) in [9]). Since these solutions are combined only from parts of the solutions we show in Table 1, such composite solutions are not presented in Table 1.

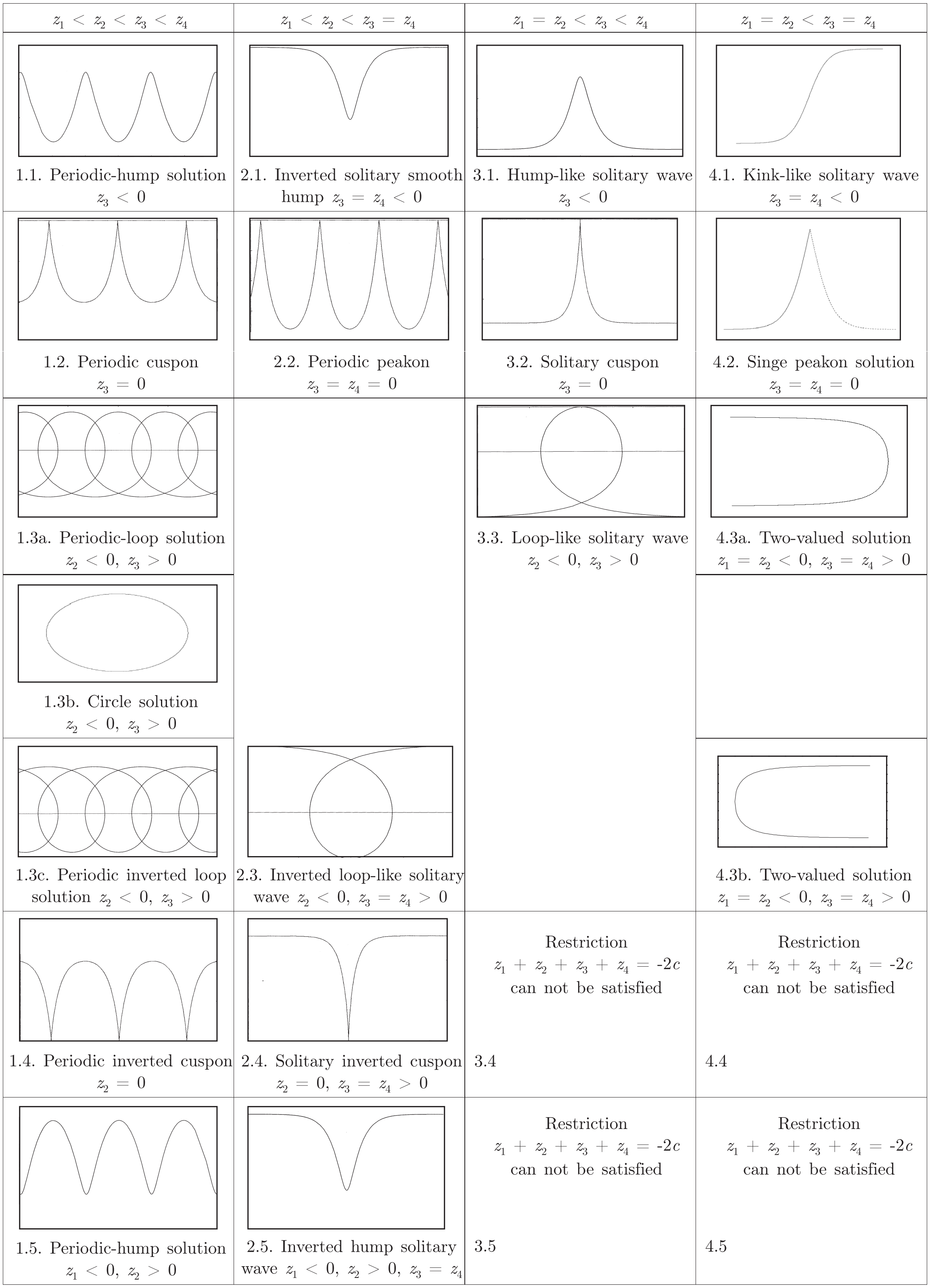
**4. The graphical interpretation of the solutions.** In this section, we suggest a graphical interpretation of the solutions from Table 1. Let us consider a 3D spiral. It is shown in the first column of Table 2. If we project the spiral perpendicularly to the spiral axis, then we will see the periodic hump given by the curve in Fig. 1.1 in Table 2. At a specific projection angle to the spiral axis, the projection of the spiral will appear as a periodic cuspon (Fig. 1.2 in Table 2). Changing the angle between the direction of observation and the axis of the spiral, we can then see a periodic-loop solution (Fig. 1.3 a in Table 2). In the exceptional case where the observation takes place along the spiral axis, the spiral appears as a circle (Fig. 1.3 b in Table 2). Thereafter the solutions are repeated in the reverse sequence: a periodic inverted loop solution (Fig. 1.3 c in Table 2), and a periodic inverted cuspon (Fig. 1.4 in Table 2), a periodic-hump solution (Fig. 1.5 in Table 2). Hence, we see all the solutions from the first column of Table 1.

To interpret the solutions from the second and third columns of Table 1, let us consider the curves in the relevant columns of Table 2. These curves comprise one loop taken from a spiral. In the second column in Table 2, the upper part of the loop is extended, whereas the lower part is extended in the third column. At different projection angles for these curves on the plane, we observe a solitary smooth hump (Fig. 2.1 in Table 2), a hump-like solitary wave (Fig. 3.1 in Table 2), a periodic peakon (Fig. 2.2 in Table 2), a solitary cuspon (Fig. 3.2 in Table 2), a loop-like solitary wave (Figs. 2.3 and 3.3 in Table 2), a solitary inverted cuspon (Fig. 2.4 in Table 2), and an inverted hump (Fig. 2.5 in Table 2).

Finally, let us consider the 3D curve which is shown in the fourth column of Table 2. It is none other than a half loop of a spiral with expanded upper and lower parts. This curve enables us to interpret the solutions from the fourth column. The projections give a kink-like solitary wave (Fig. 4.1 in Table 2), a single peakon solution (Fig. 4.2 in Table 2), and finally, a two-valued solution (Fig. 4.3 in Table 2).

Consequently, all the types of the solution from Table 1 are interpreted in Table 2.

**5. Conclusion.** A new nonlinear evolution equation generalizing the CHe and the DPe is suggested. This equation can be applied to describe shallow water waves, turbulent flows, and wave propagation in relaxing media. It can be integrated in a similar way to the CHe and the DPe in order to find travelling wave solutions. It turns out that the solutions of this new equation can be interpreted as the projection of a spiral on a plane at different projection angles to the axis of the spiral. The classification of the travelling wave solutions presented in this paper may be of help in the understanding and description of the physical processes being investigated.



$z_1 < z_2 < z_3 < z_4$
$z_1 < z_2 < z_3 = z_4$
$z_1 = z_2 < z_3 < z_4$
$z_1 = z_2 < z_3 = z_4$
1.1. Periodic-hump solution $z_3 < 0$
2.1. Inverted solitary smooth hump $z_3 = z_4 < 0$
3.1. Hump-like solitary wave $z_3 < 0$
4.1. Kink-like solitary wave $z_3 = z_4 < 0$
1.2. Periodic cuspon $z_3 = 0$
2.2. Periodic peakon $z_3 = z_4 = 0$
3.2. Solitary cuspon $z_3 = 0$
4.2. Singe peakon solution $z_3 = z_4 = 0$
1.3a. Periodic-loop solution $z_2 < 0,\ z_3 > 0$
3.3. Loop-like solitary wave $z_2 < 0,\ z_3 > 0$
4.3a. Two-valued solution $z_1 = z_2 < 0,\ z_3 = z_4 > 0$
1.3b. Circle solution $z_2 < 0,\ z_3 > 0$
1.3c. Periodic inverted loop solution $z_2 < 0,\ z_3 > 0$
2.3. Inverted loop-like solitary wave $z_2 < 0,\ z_3 = z_4 > 0$
4.3b. Two-valued solution $z_1 = z_2 < 0,\ z_3 = z_4 > 0$
1.4. Periodic inverted cuspon $z_2 = 0$
2.4. Solitary inverted cuspon $z_2 = 0,\ z_3 = z_4 > 0$
Restriction $z_1 + z_2 + z_3 + z_4 = -2c$ can not be satisfied
3.4
Restriction $z_1 + z_2 + z_3 + z_4 = -2c$ can not be satisfied
4.4
1.5. Periodic-hump solution $z_1 < 0,\ z_2 > 0$
2.5. Inverted hump solitary wave $z_1 < 0,\ z_2 > 0,\ z_3 = z_4$
Restriction $z_1 + z_2 + z_3 + z_4 = -2c$ can not be satisfied
3.5
Restriction $z_1 + z_2 + z_3 + z_4 = -2c$ can not be satisfied
4.5


Fig. 1

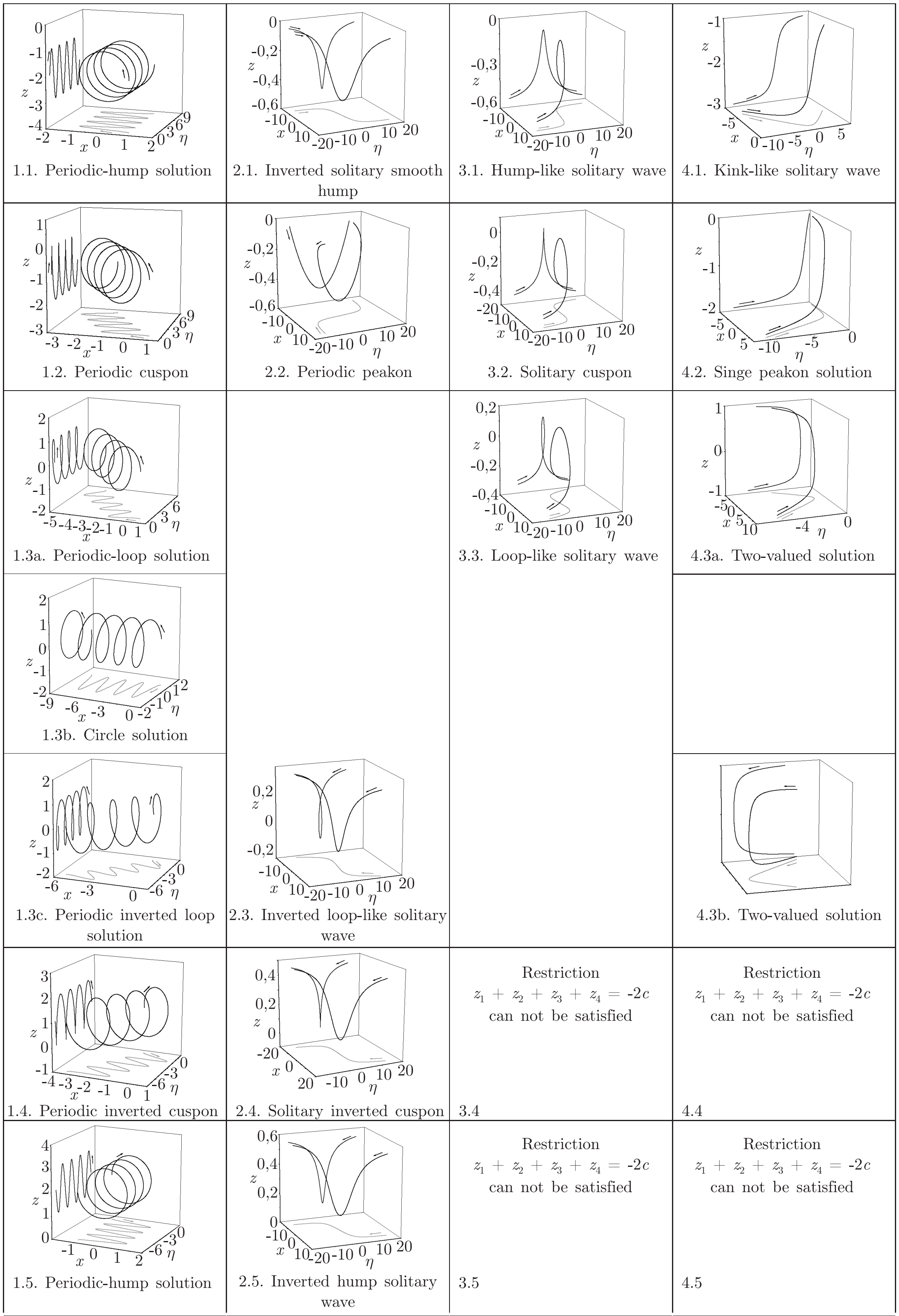

1.1. Periodic-hump solution
2.1. Inverted solitary smooth hump
3.1. Hump-like solitary wave
4.1. Kink-like solitary wave
1.2. Periodic cuspon
2.2. Periodic peakon
3.2. Solitary cuspon
4.2. Singe peakon solution
1.3a. Periodic-loop solution
3.3. Loop-like solitary wave
4.3a. Two-valued solution
1.3b. Circle solution
1.3c. Periodic inverted loop solution
2.3. Inverted loop-like solitary wave
4.3b. Two-valued solution
1.4. Periodic inverted cuspon
2.4. Solitary inverted cuspon
Restriction
$z_1 + z_2 + z_3 + z_4 = -2c$
can not be satisfied
3.4
Restriction
$z_1 + z_2 + z_3 + z_4 = -2c$
can not be satisfied
4.4
1.5. Periodic-hump solution
2.5. Inverted hump solitary wave
Restriction
$z_1 + z_2 + z_3 + z_4 = -2c$
can not be satisfied
3.5
Restriction
$z_1 + z_2 + z_3 + z_4 = -2c$
can not be satisfied
4.5

Fig. 2

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