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A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation

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Abstract

A Bäcklund transformation both in bilinear and in ordinary form for the transformed generalised Vakhnenko equation (GVE) is derived. It is shown that the equation has an infinite sequence of conservation laws. An inverse scattering problem is formulated; it has a third-order eigenvalue problem. A procedure for finding the exact N-soliton solution to the GVE via the inverse scattering method is described. The procedure is illustrated by considering the cases $N = 1$ and 2.

1. Introduction

The nonlinear propagation of deformation waves in a flexible long string is governed by the equation

$$u_{xt} + \operatorname{sgn}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\left[\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right]_{xx} = 0, \tag{1.1}$$

where u is the transverse displacement and s denotes the arc length measured along the solution curve from some fixed reference point on the string. The properties of (1.1), and in particular its loop-soliton solution, have been discussed by many authors, see [1–9] for example. Some related equations were presented and studied in [3,10–14]. Some other equations that have loop-soliton solutions are discussed in [15–17].

Recently we have introduced and studied three new equations which have loop-soliton solutions, namely the Vakhnenko equation (VE) [18–23], the generalised Vakhnenko equation (GVE) [24], and the modified generalised Vakhnenko equation (mGVE) [25]. The GVE and mGVE also have hump-soliton and cusp-soliton solutions.

The VE, namely

$$\frac{\partial}{\partial x}\mathscr{D}u + u = 0, \quad \text{where} \quad \mathscr{D} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{1.2}$$

was first presented by Vakhnenko in [18] to describe high-frequency waves in a relaxing medium [19].

In [20–23] we discussed the multi-loop soliton solution to the VE with boundary condition $u \to 0$ as $|x| \to \infty$. The key step in finding this solution is to introduce the transformations

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$$x = \theta(X,T) := T + W(X,T) + x_0, \quad t = X, \quad W = \int_{-\infty}^{X} U(X',T)\,\mathrm{d}X', \tag{1.3}$$

where x_0 is a constant, $u(x,t) = U(X,T) = W_X(X,T)$, and it is assumed that, as $|X| \to \infty$, $U \to 0$, the derivatives of W vanish, and W tends to a constant. In terms of the new variables the VE may be written

$$W_{XXT} + (1 + W_T)W_X = 0, \tag{1.4}$$

the transformed VE. Solutions in implicit form to (1.2) are found by first solving (1.4) and then transforming back to the original variables.

(1.4) may be solved by Hirota's method. This is accomplished by writing

$$W = 6(\ln f(X,T))_X \tag{1.5}$$

so that (1.4) may be written as the bilinear equation

$$(D_X^3 D_T + D_X^2) f \cdot f = 0, \tag{1.6}$$

where D is the Hirota D operator [26]. The 2 and N loop soliton solutions to (1.2) were obtained via (1.6) in [20,21] respectively.

In [23] we derived a Bäcklund transformation associated with (1.4) and hence showed that the IST problem involves a third-order eigenvalue problem; we went on to recover the N loop soliton solution to (1.2) by using the IST method.

The main aim of the present paper is to extend the investigation of the VE in [23] to the GVE [24], namely

$$\frac{\partial}{\partial x}\left(\mathscr{D}^2 u + \frac{1}{2}u^2 + \beta u\right) + \mathscr{D}u = 0 \tag{1.7}$$

or equivalently

$$\left(\frac{\partial u}{\partial x} + \mathscr{D}\right)\left(\frac{\partial}{\partial x}\mathscr{D}u + u + \beta\right) = 0,$$

where β is a real arbitrary constant.

The transformed version of the GVE (1.7) is

$$U_{XXT} + UU_T + U_X \int_{-\infty}^{X} U_T(X',T)\,\mathrm{d}X' + U_X + \beta U_T = 0, \tag{1.8}$$

or equivalently

$$W_{XXT} + (1 + W_T)W_X + \beta W_T = 0. \tag{1.9}$$

The corresponding bilinear equation is

$$F(D_X, D_T) f \cdot f = 0, \tag{1.10}$$

where

$$F(D_X, D_T) := D_X^3 D_T + D_X^2 + \beta D_X D_T. \tag{1.11}$$

Obviously, with $\beta = 0$, (1.9) and (1.10) reduce to Eqs. (1.4) and (1.6) respectively which are associated with the VE. With $\beta = -1$ and $T \to -T$, (1.9) and (1.10) are associated with the Hirota–Satsuma equation (HSE) for shallow water waves [26,27]. The solution to the HSE by Hirota's method is given in [27]. It follows that $F(D_X, D_T)$, as given by (1.11) with $\beta = -1$ and $T \to -T$, satisfies the 'N-soliton condition' (NSC) [26]. When $\beta < 0$, the scalings

$$T \to -\mu T, \quad X \to X/\mu, \quad \text{where} \quad \mu = \sqrt{-\beta}, \tag{1.12}$$

transform (1.10) into the bilinear form of the HSE. However this is not possible when $\beta > 0$. In [24] we showed that the NSC does in fact hold for arbitrary nonzero β and went on to find the N-soliton solution to (1.7) via use of Hirota's method applied to (1.10). A novel feature of the GVE is that different types of soliton solutions are possible, namely hump-like, cusp-like or loop-like.

As far as we are aware, the solution by the IST method to the HSE (i.e. Eq. (1.8) with $\beta = -1$ and $T \to -T$) has not been given explicitly in the literature. In this paper we present the IST method to solve (1.8) for arbitrary nonzero β and hence find the N-soliton solution to (1.7) subject to the boundary condition $u \to 0$ as $|x| \to \infty$.

In Section 2 we derive the Bäcklund transformation and Lax pair for the transformed GVE. It is found that the IST problem for the transformed GVE has a third-order eigenvalue problem. We also show that there is an infinite sequence of conservation laws associated with the transformed GVE. In Section 3 we use the IST method to find the N-soliton solution of the GVE. Finally, in Section 4, we consider the one-soliton and two-soliton solutions in more detail and show that the solutions correspond to those derived via Hirota's method in [24].

2. Bäcklund transformation, Lax pair and conservation laws for the transformed GVE

We will show that the Bäcklund transformation for (1.10) is given by the two equations

$$(D_X^3 + \beta D_X - \lambda(X))f' \cdot f = 0, \tag{2.1}$$

$$(3D_X D_T + 1 + \mu(T)D_X)f' \cdot f = 0, \tag{2.2}$$

where $\lambda(X)$ is an arbitrary function of X and $\mu(T)$ is an arbitrary function of T. We follow the method developed in [28].

Consider the expression P defined by

$$P := [(D_T D_X^3 + D_X^2 + \beta D_X D_T)f' \cdot f']ff - f'f'[(D_T D_X^3 + D_X^2 + \beta D_X D_T)f \cdot f], \tag{2.3}$$

where $f \neq f'$. In [23] it was shown that

$$[(D_T D_X^3 + D_X^2)f' \cdot f']ff - f'f'[(D_T D_X^3 + D_X^2)f \cdot f] = 2D_T(D_X^3 f' \cdot f) \cdot (f'f) - 2D_X(\{3D_T D_X + 1\}f' \cdot f) \cdot (D_X f' \cdot f). \tag{2.4}$$

By using (2.4) and the identities (II.1) and (VII.2) from [29], P given by (2.3) can be reduced to the form

$$P = 2D_T(\{D_X^3 + \beta D_X - \lambda(X)\}f' \cdot f) \cdot (f'f) - 2D_X(\{3D_T D_X + 1 + \mu(T)D_X\}f' \cdot f) \cdot (D_X f' \cdot f). \tag{2.5}$$

It is clear from (2.5) that if (2.1) and (2.2) hold then $P = 0$. Furthermore it then follows from (2.3) that if f is a solution of (1.10) then so is f' and vice-versa. Consequently, we have proved that the two Eqs. (2.1) and (2.2) constitute a Bäcklund transformation for Eq. (1.10). As expected, with $\beta = -1$ and $T \to -T$, (2.1) and (2.2) become the Bäcklund transformation for the HSE (see Eqs. (5.131) and (5.132) in [26]).

The inclusion of μ in the operator $3D_T + \mu(T)$ which appears in (2.2) corresponds to a multiplication of f and f' by terms of the form $e^{g(T)}$ and $e^{g'(T)}$ respectively; from (1.5) we see that this has no effect on W or W'. Hence, without loss of generality, we may take $\mu = 0$ in (2.2) if we wish.

By introducing the function

$$\psi = f'/f, \tag{2.6}$$

and taking into account (1.5), we find that (2.1) and (2.2) reduce to

$$\psi_{XXX} + (\beta + W_X)\psi_X - \lambda\psi = 0, \tag{2.7}$$

$$3\psi_{XT} + (1 + W_T)\psi + \mu\psi_X = 0 \tag{2.8}$$

respectively, where we have used results similar to (X.1)–(X.3) in [26].

From (2.7) and (2.8) it can be shown that

$$3\lambda\psi_T + (1 + W_T)\psi_{XX} - W_{XT}\psi_X + [W_{XXT} + (\beta + W_X)(1 + W_T) + \mu\lambda]\psi = 0 \tag{2.9}$$

and

$$[W_{XXT} + (1 + W_T)W_X + \beta W_T]_X\psi + (3\psi_T + \mu\psi)\lambda_X = 0. \tag{2.10}$$

In view of (1.9), (2.9) becomes

$$3\lambda\psi_T + (1 + W_T)\psi_{XX} - W_{XT}\psi_X + (\beta + \lambda\mu)\psi = 0, \tag{2.11}$$

and (2.10) implies that $\lambda_X = 0$ so the spectrum λ of (2.7) remains constant. Constant λ is what is required in the IST problem discussed in Section 3. (2.7) and (2.11) are the Lax pair for (1.9). As expected, with $\beta = -1$ and $T \to -T$, (2.7), (2.8) and (2.11) are the corresponding equations for the HSE (cf. Eqs. (A8a), (A8b) and (A10b) respectively in [30]).

Following the procedure given in [26,31], we can rewrite (2.7) and (2.11) in terms of the potential W. Recalling that $\psi = f'/f$, and noting that $W' - W = 6\varphi_X$ and $W' + W = 6\rho_X$, where $\varphi = \ln f'/f$ and $\rho = \ln f'f$, we find that (2.7) and (2.11) give the following Bäcklund transformation in ordinary form:

$$(W'-W)_{XX} + \tfrac{1}{2}(W'-W)(W'+W)_X + \tfrac{1}{36}(W'-W)^3 + \beta(W'-W) - 6\lambda = 0, \tag{2.12}$$

$$3\lambda(W'-W)_T + \Big[(1+W_T)((W'-W)_X + \tfrac{1}{6}(W'-W)^2) - W_{XT}(W'-W)\Big]_X = 0. \tag{2.13}$$

A method for deriving higher conservation laws via the Bäcklund transformation was given in [32,33]. The method was applied to a higher-order KdV equation in [31]. Apart from a scaling factor and the term involving β, our (2.12) is the same as (29) in [31], and our (2.13) is in conservation form like (30) in [31]. Not surprisingly, when the method is applied to the transformed GVE in the form (1.8), we obtain conserved densities which, apart from scaling factors and terms involving β, agree with those given by Eqs. (37)–(39) in [31]. We also deduce that an infinite sequence of conservation laws is associated with (1.8) and that, for example, the first two nontrivial conserved densities are U and $(U^3 - 3U_X^2 + 3\beta U^2)$.

3. The IST problem and its N-soliton solution

As shown in Section 2, the IST problem for the transformed GVE (1.9) has a spectral equation for ψ of third-order, namely (2.7). The inverse problem for certain third-order spectral equations has been considered by Kaup [34] and Caudrey [35,36]. The time evolution of ψ is determined from (2.8) or (2.11).

Following the method described by Caudrey [35], the spectral equation (2.7) can be rewritten

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X,\zeta)]\cdot\boldsymbol{\psi} \tag{3.1}$$

with

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -\beta & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -W_X & 0 \end{pmatrix}. \tag{3.2}$$

We find the eigenvalue $\lambda_j(\zeta)$ of the matrix $\mathbf{A}$ from the equation

$$\det(\mathbf{A} - \lambda_j\mathbf{E}) = -\lambda_j^3 - \beta\lambda_j + \lambda = 0, \tag{3.3}$$

where $\mathbf{E}$ is the identity matrix. The relation (3.3) between the values λ and λ_j can be rewritten in parametric form with ζ as parameter, namely

$$\lambda_j = \left(\frac{\beta}{3}\right)^{1/2}\left(\omega_j\zeta - \frac{1}{\omega_j\zeta}\right), \tag{3.4}$$

$$\lambda = \left(\frac{\beta}{3}\right)^{3/2}\left(\zeta^3 - \frac{1}{\zeta^3}\right), \tag{3.5}$$

where $\omega_j = e^{i2\pi(j-1)/3}$ are the cube of roots of 1 $(j = 1,2,3)$. Because of the properties $\lambda_1(\zeta) = \lambda_1(-\zeta^{-1})$, $\lambda_2(\zeta) = \lambda_3(-\zeta^{-1})$, $\lambda_3(\zeta) = \lambda_2(-\zeta^{-1})$ and $\lambda(\zeta) = \lambda(-\zeta^{-1})$, it is sufficient to consider the values ζ located outside (or inside) of the circle $|\zeta| = 1$ only.

The right- and left-eigenvectors are

$$\mathbf{v}_j(\zeta) = \begin{pmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \end{pmatrix}, \quad \tilde{\mathbf{v}}_j(\zeta) = \left(\lambda_j^2 + \beta, \lambda_j, 1\right). \tag{3.6}$$

It should be noted that the passage to the limit $\beta \to 0$ must be carried out with $\sqrt{\beta}\zeta$ held constant.

The general theory of the inverse scattering problem for N spectral equations has been developed in [35]. The solution of the linear equation (3.1) (or equivalently (1.9)) has been obtained by Caudrey [35] in terms of Jost functions $\boldsymbol{\phi}_j(X,\zeta)$ which have the asymptotic behaviour

$$\mathbf{\Phi}_j(X,\zeta) := \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X,\zeta) \to \mathbf{v}_j(\zeta) \quad \text{as } X \to -\infty. \tag{3.7}$$

Here T is regarded as a parameter until the T-evolution of the scattering data is taken into account later. The solution of the direct problem is given by the equation system (4.5) in [35]. We shall restrict our attention to the N-soliton solution. To do this we consider Eq. (6.20) from [35] by putting $Q_{ij}(\zeta) \equiv 0$. Then there is only the bound state spectrum which is associated with the soliton solutions.

Let the bound state spectrum be defined by K poles located, for definiteness, outside the circle $|\zeta| = 1$. The relation (4.5) from [35] is reduced to the form

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{K}\sum_{j=2}^{3}\gamma_{1j}^{(k)}\frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)}\Phi_1(X,\omega_j\zeta_1^{(k)}). \tag{3.8}$$

We need only consider the function $\Phi_1(X,\zeta)$ since there is a set of symmetry properties as for the Boussinesq equation, namely the properties (6.15), (6.16) in [35] for Jost functions $\boldsymbol{\phi}_j(X,\zeta)$

$$\phi_1(X,\zeta/\omega_1) = \phi_2(X,\zeta/\omega_2) = \phi_3(X,\zeta/\omega_3), \quad \boldsymbol{\phi}_j(X,\zeta) = \boldsymbol{\phi}_j(X,-\zeta^{-1}) \tag{3.9}$$

that follow from Eqs. (3.2) and (3.4).

(3.8) involves the spectral data, namely the poles $\zeta_i^{(k)}$ and the quantities $\gamma_{ij}^{(k)}$. First we will prove that $\mathrm{Re}\lambda = 0$ for compact support. Indeed, from (2.7) we have

$$(\psi_X)_{XXX} + [(\beta + W_X)\psi_X]_X - \lambda\psi_X = 0, \tag{3.10}$$

and together with (2.8) this enables us to write

$$\frac{\partial}{\partial X}\left(\frac{\partial^2}{\partial X^2}\psi_X\psi^* - 3\psi_{XX}\psi_X^* + (\beta + W_X)\psi_X\psi^*\right) - 2\mathrm{Re}\lambda\psi_X\psi^* = 0. \tag{3.11}$$

Integrating (3.11) over all values of X, we obtain that for compact support $\mathrm{Re}\lambda = 0$ since, in the general case, $\int_{-\infty}^{\infty}\psi_X\psi^*\,\mathrm{d}X \neq 0$.

As follows from Eqs. (2.12), (2.13), (2.36) and (2.37) of [34], $\psi_X(\zeta)$ is related to the adjoint states $\psi^A(-\zeta)$. In the usual manner, using the adjoint states and Eq. (14) from [36], and Eq. (2.37) from [34], one can obtain

$$\phi_{1X}(X,\zeta) = \frac{\mathrm{i}}{\sqrt{3}}[\phi_{1X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) - \phi_{1X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)]. \tag{3.12}$$

It is easily seen that if $\zeta_1^{(1)}$ is a pole of $\phi_1(X,\zeta)$, then there is a pole either at $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_2\zeta)$ has a pole), or at $\zeta_1^{(2)} = -\omega_3\zeta_1^{(1)}$ (if $\phi_1(X,-\omega_3\zeta)$ has a pole). For definiteness, let $\zeta_1^{(2)} = -\omega_2\zeta_1^{(1)}$, then as follows from Eq. (3.12) the point $-\omega_3\zeta_1^{(2)}$ should be a pole. However, this pole coincides with the pole $\zeta_1^{(1)}$, since $-\omega_3\zeta_1^{(2)} = -\omega_3(-\omega_2)\zeta_1^{(1)} = \zeta_1^{(1)}$. Hence, the poles appear in pairs $\zeta_1^{(2n-1)}$, $\zeta_1^{(2n)}$ under the condition $\zeta_1^{(2n)}/\zeta_1^{(2n-1)} = -\omega_2$, where n is the number pair.

Let us consider N pairs of poles, i.e. in all there are $K = 2N$ poles over which the sum is taken in (3.8). For the pair n ($n = 1, 2, \ldots, N$) we have the properties

$$\text{(i)} \quad \zeta_1^{(2n-1)} = \mathrm{i}\omega_2\xi_n, \qquad \text{(ii)} \quad \zeta_1^{(2n)} = -\mathrm{i}\omega_3\xi_n. \tag{3.13}$$

Since U is real and λ is imaginary, either ξ_n is real when $\beta > 0$ or ξ_n is imaginary when $\beta < 0$, i.e. $\sqrt{\beta}\xi_n$ is real.

By considering Eq. (3.12) in the vicinity of the first pole $\zeta_1^{(2n-1)}$ of the pair n and using the relation (3.8), one can obtain a relation between $\gamma_{12}^{(2n-1)}$ and $\gamma_{13}^{(2n)}$. In this case the functions $\phi_{1X}(X,\zeta)$, $\phi_1(X,-\omega_2\zeta)$, $\phi_{1X}(X,-\omega_2\zeta)$ also have poles here, while the functions $\phi_1(X,-\omega_3\zeta)$, $\phi_{1X}(X,-\omega_3\zeta)$ do not have poles here. Substituting $\phi_1(X,\zeta)$ in the form given by (3.7) and (3.8) into Eq. (3.12) and letting $X \to -\infty$, we have $\gamma_{12}^{(2n)} = \gamma_{13}^{(2n-1)} = 0$ and the ratio

$$\frac{\gamma_{12}^{(2n-1)}}{\gamma_{13}^{(2n)}} = \frac{\omega_2\xi_n + (\omega_2\xi_n)^{-1}}{\omega_3\xi_n + (\omega_3\xi_n)^{-1}}. \tag{3.14}$$

Therefore the properties of $\gamma_{ij}^{(k)}$ should be defined by the relationships

$$\left.\begin{array}{lll} \text{(i)} & \gamma_{12}^{(2n-1)} = \sqrt{\beta}\gamma_n[\omega_2\xi_n + (\omega_2\xi_n)^{-1}], & \gamma_{13}^{(2n-1)} = 0, \\ \text{(ii)} & \gamma_{12}^{(2n)} = 0, & \gamma_{13}^{(2n)} = \sqrt{\beta}\gamma_n[\omega_3\xi_n + (\omega_3\xi_n)^{-1}], \end{array}\right\} \tag{3.15}$$

where γ_n are arbitrary constants. We will show below that γ_n is real when W_X is real.

Following [23] we expand $\Phi_1(X,\zeta)$ as an asymptotic series in $\lambda_1^{-1}(\zeta)$ to obtain

$$\Phi_1(X,\zeta) = 1 - \frac{1}{3\lambda_1(\zeta)}[W(X) - W(-\infty)] + \mathrm{O}(\lambda_1^{-2}(\zeta)). \tag{3.16}$$

On the other hand, we may rewrite the relationship (3.8) as (see, for instance, Eqs. (6.33) and (6.34) in [35])

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{K} \frac{\exp\{-\lambda_1(\zeta_1^{(k)})X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Psi_k(X), \tag{3.17}$$

$$\Psi_k(X) = \sum_{j=2}^{3} \gamma_{1j}^{(k)} \exp\{\lambda_j(\zeta_1^{(k)})X\}\Phi_1(X,\omega_j\zeta_1^{(k)}).$$

From (3.16) and (3.17) it may be shown that (cf. Eq. (6.38) in [35])

$$W(X) - W(-\infty) = -3\sum_{k=1}^{K} \exp\{-\lambda_1(\zeta_1^{(k)})X\}\Psi_k(X) = 3\frac{\partial}{\partial X}\ln(\det M). \tag{3.18}$$

The matrix M is defined as in the relationship (6.36) in [35] by

$$M_{kl}(X) = \delta_{kl} - \sum_{j=2}^{3} \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})]X\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})}. \tag{3.19}$$

Now let us consider the T-evolution of the spectral data. By analyzing the solution of Eq. (2.8) when $X \to -\infty$ together with (3.7), we find that

$$\phi_i(X,T,\zeta) = \exp\Big[-(3\lambda_i(\zeta))^{-1}T\Big]\phi_i(X,0,\zeta).$$

Hence the T-evolution of the scattering data is given by the relationships (with $k = 1,2,\ldots,K$)

$$\left.\begin{aligned} &\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0),\\ &\gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0)\exp\left\{\left[-\Big(3\lambda_j(\zeta_1^{(k)})\Big)^{-1} + \Big(3\lambda_1(\zeta_1^{(k)})\Big)^{-1}\right]T\right\} \end{aligned}\right\}. \tag{3.20}$$

The final result, including the T-evolution, for the N-soliton solution of the transformed GVE (1.8) is

$$U(X,T) = W_X(X,T) = 3\frac{\partial^2}{\partial X^2}\ln(\det M(X,T)), \tag{3.21}$$

where M is the $2N \times 2N$ matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}(0)\frac{\exp\left\{\left[-\Big(3\lambda_j(\zeta_1^{(k)})\Big)^{-1} + \Big(3\lambda_1(\zeta_1^{(k)})\Big)^{-1}\right]T + \Big(\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})\Big)X\right\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})} \tag{3.22}$$

and

$$n = 1,2,\ldots,N, \quad m = 2n-1,$$

$$\lambda_1(\zeta_1^{(m)}) = \mathrm{i}\sqrt{\beta/3}[\omega_2\xi_m + (\omega_2\xi_m)^{-1}], \quad \lambda_2(\zeta_1^{(m)}) = \mathrm{i}\sqrt{\beta/3}[\omega_3\xi_m + (\omega_3\xi_m)^{-1}],$$

$$\gamma_{12}^{(m)}(0) = \sqrt{\beta}\gamma_m(0)[\omega_2\xi_m + (\omega_2\xi_m)^{-1}], \quad \gamma_{13}^{(m)} = 0,$$

$$\lambda_1(\zeta_1^{(m+1)}) = -\mathrm{i}\sqrt{\beta/3}[\omega_3\xi_m + (\omega_3\xi_m)^{-1}], \quad \lambda_3(\zeta_1^{(m+1)}) = -\mathrm{i}\sqrt{\beta/3}[\omega_2\xi_m + (\omega_2\xi_m)^{-1}],$$

$$\gamma_{12}^{(m+1)} = 0, \quad \gamma_{13}^{(m+1)}(0) = \sqrt{\beta}\gamma_m(0)[\omega_3\xi_m + (\omega_3\xi_m)^{-1}].$$

For the N-soliton solution there are N arbitrary constants ξ_m and N arbitrary constants γ_m. We note that comparison of (1.5) with (3.21) shows that

$$\ln(\det M) = 2\ln f \tag{3.23}$$

so that $\det M$ should be a perfect square for arbitrary N.

Finally, the N-soliton solution of the untransformed GVE (1.7) is given in parametric form by

$$u(x,t) = U(t,T), \quad x = \theta(t,T), \tag{3.24}$$

where $\theta(X,T)$ is defined in (1.3).

4. Examples of one- and two-soliton solutions

In order to obtain the one-soliton solution of the transformed GVE (1.8) we need first to calculate the 2×2 matrix M according to (3.22) with $N = 1$. The elements of the matrix are

$$\begin{aligned}
M_{11} &= 1 - \frac{\sqrt{\beta}\gamma_1}{2k}[\omega_2\xi_1 + (\omega_2\xi_1)^{-1}]\exp[2k(X - cT)],\\
M_{12} &= \frac{\sqrt{3}\gamma_1 \mathrm{i}}{2}\left\{\frac{[\omega_2\xi_1 + (\omega_2\xi_1)^{-1}]}{[\omega_3\xi_1 + (\omega_3\xi_1)^{-1}]}\right\}\exp\left\{2\mathrm{i}\sqrt{\beta/3}[\omega_3\xi_1 + (\omega_3\xi_1)^{-1}]X - 2kcT\right\},\\
M_{21} &= -\frac{\sqrt{3}\gamma_1 \mathrm{i}}{2}\left\{\frac{[\omega_3\xi_1 + (\omega_3\xi_1)^{-1}]}{[\omega_2\xi_1 + (\omega_2\xi_1)^{-1}]}\right\}\exp\left\{-2\mathrm{i}\sqrt{\beta/3}[\omega_2\xi_1 + (\omega_2\xi_1)^{-1}]X - 2kcT\right\},\\
M_{22} &= 1 - \frac{\sqrt{\beta}\gamma_1}{2k}[\omega_3\xi_1 + (\omega_3\xi_1)^{-1}]\exp[2k(X - cT)]
\end{aligned} \tag{4.1}$$

and the determinant of the matrix is

$$\det M = \left\{1 + \frac{\gamma_1}{2}\left(\frac{\xi_1^2 + 1}{\xi_1^2 - 1}\right)\exp[2k(X - cT)]\right\}^2, \tag{4.2}$$

where $k = \sqrt{\beta}(\xi_1 - \xi_1^{-1})/2$ and $c^{-1} = \beta(\xi_1^2 + \xi_1^{-2} - 1)$. Notice that this determinant is a perfect square; this is consistent with (3.23).

From (3.21) and (4.2), the one-soliton solution of the transformed GVE (1.8), as obtained by the IST method, is

$$U(X,T) = 6k^2\operatorname{sech}^2[k(X - cT) + \alpha_1], \tag{4.3}$$

where

$$\alpha_1 = \frac{1}{2}\ln\left[\frac{\gamma_1}{2}\left(\frac{\xi_1^2 + 1}{\xi_1^2 - 1}\right)\right].$$

α_1 is an arbitrary constant. Since U is real, it follows from (4.3) that α_1 is real; moreover, since $\sqrt{\beta}\xi_1$ is real, γ_1 is also real. (4.3) agrees with the one-soliton solution to the transformed GVE as found by Hirota's method and given by (4.1)–(4.4) in [24].

In a similar way (details omitted) we find that for the two-soliton solution M is a 4×4 matrix for which $\det M$ is a perfect square given by

$$\det M = \left(1 + q_1^2 + q_2^2 + b_{12}^2 q_1^2 q_2^2\right)^2, \tag{4.4}$$

where

$$q_i = \exp[2k_i(X - c_iT) + \alpha_i], \tag{4.5}$$

$$b_{12}^2 = \left(\frac{\xi_3 - \xi_3^{-1} - \xi_1 + \xi_1^{-1}}{\xi_3 - \xi_3^{-1} + \xi_1 - \xi_1^{-1}}\right)^3 \frac{\xi_3^3 - \xi_3^{-3} + \xi_1^3 - \xi_1^{-3}}{\xi_3^3 - \xi_3^{-3} - \xi_1^3 + \xi_1^{-3}}, \tag{4.6}$$

$$k_i = \sqrt{\beta}(\xi_{2i-1} - \xi_{2i-1}^{-1})/2, \quad c_i^{-1} = \beta(\xi_{2i-1}^2 + \xi_{2i-1}^{-2} - 1),$$

$$\alpha_i = \frac{1}{2}\ln\left[\frac{\gamma_{2i-1}}{2}\left(\frac{\xi_{2i-1}^2 + 1}{\xi_{2i-1}^2 - 1}\right)\right].$$

The α_i are real arbitrary constants. The relationship (3.21) together with (4.4) gives the two-soliton solution of (1.8). (4.4)–(4.6) agree with the two-soliton solution as found by Hirota's method and given by (7.1)–(7.7) in [24].

In passing we note that in the limit $\beta \to 0$ with $\sqrt{\beta}\xi_i$ held constant, the one- and two-soliton solutions given above reduce to the ones obtained in [23] for the VE.

By combining the above results with (3.24) we obtain the one- and two-soliton solutions to the GVE. As discussed in detail in [24] the shape of the one-soliton solution to the GVE (1.7) depends on the value of $q := \beta/k^2$. Examples of

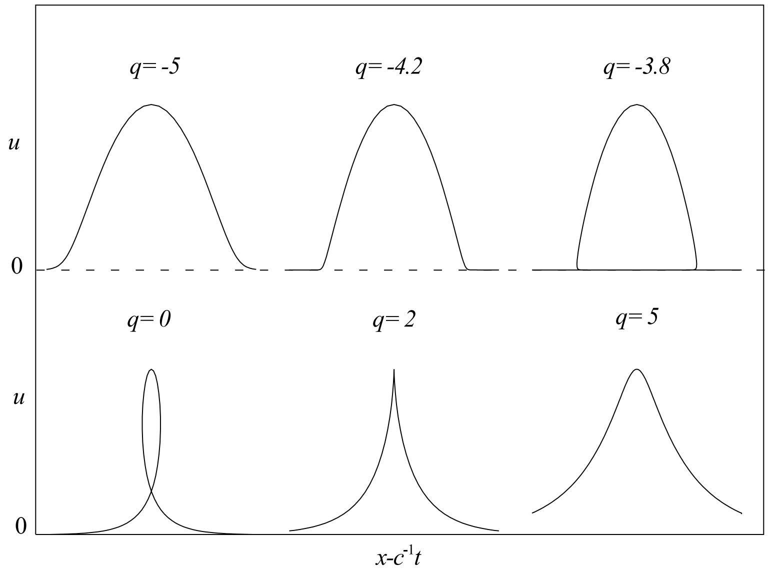

Fig. 1. Hump-like $u(x,t)$ for $q = -5$, -4.2 and 5; loop-like $u(x,t)$ for $q = -3.8$ and 0, and the cusp-shaped $u(x,t)$ for $q = 2$.

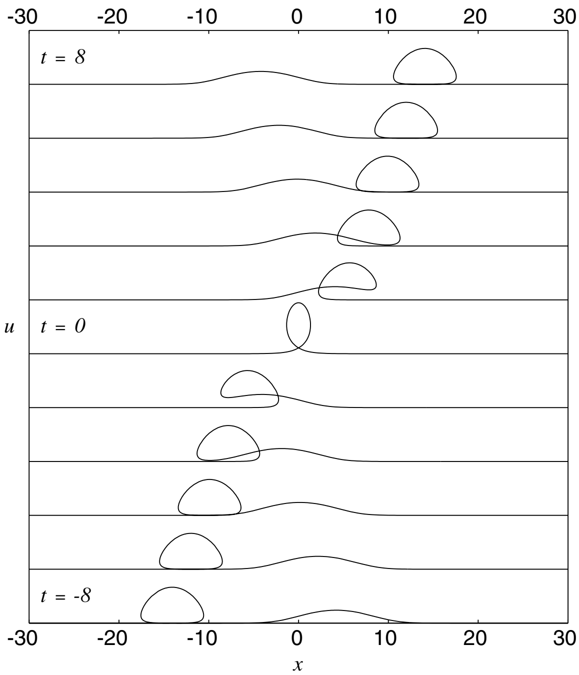

Fig. 2. The interaction process for $r = 0.6$ and $s = -2.7$.

loop-like and hump-like solitons, and the cusp-shaped soliton, are illustrated in Fig. 1 where u is plotted against $x - c^{-1}t$. The interaction process between the solitons in the two-soliton solution to the GVE was also discussed in detail in [24]. The character of the interaction depends on the values of $r := k_1/k_2$ and $s := \beta/k_2^2$. An example of an interaction is given in Fig. 2 where u is plotted against x at several equally spaced values of t.

5. Conclusion

We have extended our work on the VE (1.2) presented in [23] to the GVE (1.7). In particular in Section 3 we found the N-soliton solution of the transformed GVE by using the IST method. This result, together with (3.24), gives the N-soliton solution to the GVE. In principle this solution is equivalent to the one in [24] as derived by Hirota's method. In Section 4 we showed that the one- and two-soliton solutions to the GVE discussed in detail in [24] are recovered explicitly.

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