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THE VAKHNENKO EQUATION FROM THE VIEWPOINT OF THE INVERSE SCATTERING METHOD FOR THE KdV EQUATION

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Abstract: The formulation of the inverse scattering transform method is discussed for the nonlinear evolution equation $(u_t + uu_x)_x + u = 0$ (the Vakhnenko equation). It is shown that the equation system for the inverse scattering problem associated with the Vakhnenko equation can not contain the isospectral Schrödinger equation. The exact two soliton solutions are obtained by means of the use of elements of the inverse scattering problem for the KdV equation.

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1. Introduction

This paper deals with a nonlinear evolution equation (see [1])

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0\,, \tag{1.1}$$

that was proposed by Vakhnenko. The equation arose as a result describ-

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ing high-frequency perturbations in a relaxing medium [2]. In contrast to the high-frequency perturbations, the low-frequency perturbations satisfy the well-known KdV equation [1] that is widely encountered in applications [3]. Eq. (1.1) has been studied in various Refs. [1], [4], [5], [6], [7]. Hereafter, as was initiated in [4], this equation is referred as Vakhnenko equation (VE). It is noted that Eq. (1.1) and the KdV equation have the same hydrodynamic nonlinearity and do not contain dissipative terms; only the dispersive terms are different. The similarity between these equations indicates that, in studying the VE, the application of the inverse scattering transform (IST) method should be possible. The IST method is the most appropriate way of tackling initial value problems. The results of applying the IST method would be useful in solving the Cauchy problem for the VE. The study of the VE is of scientific interest both from the viewpoint of the existence of stable wave formations and from the viewpoint of the general problem of integrability of nonlinear equations.

In papers [1], [4] two families of periodic solutions were obtained for the VE. This was achieved by assuming travelling wave solutions and the use of direct integration. There is also a solitary wave solution that has loop like form (see Fig. 1 in [1]). This multiple-valued solution is similar to the loop-like soliton solution for the equation describing the dynamics of a stretched rope [8]. Loop-like solitons on vortex filaments have been studied in [9], [10].

We have succeeded in finding new coordinates in terms of which the solitary wave solution is given by single-valued parametric relations. The transformation into these coordinates is the key to solving the problem of the interaction of solitons. In this paper the exact two-soliton solution is obtained for the VE. It should be noted that the first attempt to obtain the interaction of the two solitons for the VE fared poorly [1]; a mistake was made in deriving the linear equation (12) in [1] which rendered the conclusions erroneous. Unlike [5], [6] where the interaction of the solitons was studied by the Hirota method [11], [12], we use now elements of the IST method for the KdV equation. The analysis of the two-soliton solution in the framework of the IST method for the KdV equation leads us to the conclusion that, for the IST problem, the equation system associated with the VE (1.1) does not contain the isospectral Schrödinger equation.

2. The Transformed VE

As previously [5], let us define new independent variables (T, X) by the transformation

$$\varphi\, dT = dx - u\, dt, \qquad X = t. \tag{2.1}$$

The function φ is to be obtained. It is important that the functions $x = \theta(T, X)$ and $u = U(T, X)$ turn out to be single-valued. In terms of the coordinates (T, X) the solution of the VE is given by single-valued parametric relations. The transformation into these coordinates is the key-point in solving the problem of the interaction of solitons as well as explaining the multiple-valued solutions [2]. The transformation (2.1) is similar to the transformation between Eulerian coordinates (x, t) and Lagrangian coordinates (T, X). We require that $T = x$ if there is no perturbation, i.e. if $u(x, t) = 0$. Hence $\varphi = 1$ when $u(x, t) = 0$. For example, it may be shown, that Eqs. (12) and (14) imply $\varphi = 1 - u/v$, for the one loop soliton solution (see [1]).

The function φ is the additional dependent variable in the equation system (2.3), (2.4) to which we reduce the original Eq. (1.1). We note that the transformation inverse to (2.1) is

$$dx = \varphi\, dT + U\, dX, \qquad U(T, X) \equiv u(x, t), \tag{2.2}$$

and taking into account the condition that dx is an exact differential, we obtain

$$\frac{\partial \varphi}{\partial X} = \frac{\partial U}{\partial T}. \tag{2.3}$$

This equation, together with Eq. (1.1) rewritten in terms of $\varphi(T, X)$ and $U(T, X)$, namely

$$\frac{\partial^2 \varphi}{\partial X^2} + U\varphi = 0, \tag{2.4}$$

is the main system of equations. The equation system (2.3), (2.4) can be reduced to a nonlinear equation in one unknown W defined by

$$W_X = U. \tag{2.5}$$

As in [5], [6], we study solutions U that vanish as $|X| \to \infty$ or, equivalently, solutions for which W tends to a constant as $|X| \to \infty$. From (2.3) and (2.5) and the requirement that $\varphi \to 1$ as $|X| \to \infty$ we have

$\varphi = 1 + W_T$; then from (2.4) we arrive at the transformed form of the VE

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{2.6}$$

Furthermore it follows from (2.2) that the original independent space coordinate x is given by

$$x = \theta(T, X) := x_0 + T + W, \tag{2.7}$$

where x_0 is an arbitrary constant. Since the functions $\theta(T, X)$ and $U(T, X)$ are single-valued, the problem of multi-valued solutions has been solved from the mathematical point of view.

3. One-soliton Solutions as Reflectionless Potentials

The method of the inverse scattering transform (IST) is a powerful method to as a means for solving nonlinear evolution equations. Let us remember that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3.1}$$

is associated with the system of the equations

$$\psi_{xx} + u\psi = \lambda\psi, \tag{3.2}$$

$$\psi_t + 3\lambda\psi_x + \psi_{xxx} + 3u\psi_x = 0. \tag{3.3}$$

The equation system (3.2), (3.3) is a case of the IST method presented in a classic paper [13]. Since the system (3.2), (3.3) contains the Schrödinger equation (3.2), we will use elements of the IST method as applied to the KdV equation in order to analyze the transformed VE. The known one-soliton solution of the KdV equation (3.1) has the form (without the time-dependence)

$$u = 2\eta^2 \operatorname{sech}^2 \eta x. \tag{3.4}$$

Here, as an example, we will consider the case $\eta = 1$.

The results in this paper are based on the assumption that the system of equations associated with (2.6), which are analogous to (3.2) and (3.3), are unknown. However, after the calculations in this paper were completed, we made some progress in the formulation of these equations, i.e. the eigenvalue problem for the VE [14].

Now let us focus on the fact that Eq. (2.4) is the Schrödinger equation

$$\frac{\partial^2\psi}{\partial X^2} - Q\psi = \lambda\psi$$

with eigenvalue (energy) $\lambda = 0$ and potential $Q = -U$. Eq. (2.4) determines the dependence on the coordinate X, and time T appears here as a parameter. However, the time-dependence is determined by Eq. (2.3).

The known one-soliton solution of Eq. (1.1), which we obtained recently [1], [4], has the form

$$U = \frac{3}{2}v\ \mathrm{sech}^2\,\frac{T - vX}{2\sqrt{v}}. \tag{3.5}$$

If it is not otherwise noted, for convenience we will consider $v = 4,\ T = 0$, and then Eq. (3.5) reduces to

$$U = 6\ \mathrm{sech}^2 X. \tag{3.6}$$

The principal fact is that both $u = 2\ \mathrm{sech}^2 x$ (3.4) and $U = 6\ \mathrm{sech}^2 X$ (3.6) relate to reflectionless potentials. The general form of the reflectionless potentials is (see Section 2.4 in [3])

$$u = m(m+1)\ \mathrm{sech}^2 x. \tag{3.7}$$

We have $m = 1$ for the potential (3.4) and $m = 2$ for the potential (3.6). It is known [3], [15] that for integrable nonlinear equations, reflectionless potentials generate soliton solutions (in the general case, N–soliton solutions).

4. Two-level Reflectionless Potential

Let us consider the one-soliton solution of the system (2.3), (2.4) in the framework of the IST method for the KdV equation. For this purpose let us analyze the Schödinger equation with the potential $Q \equiv -U = -6\ \mathrm{sech}^2 X$ (T is a parameter)

$$\frac{d^2\psi}{dX^2} - Q\psi = -k^2\psi, \qquad k^2 = -\lambda. \tag{4.1}$$

For the scattering problem, the solution of Eq. (4.1) should satisfy the boundary conditions

$$\psi(X,k) \sim \begin{cases} e^{-ikX}, & X \to -\infty, \\ b(k)\,e^{ikX} + a(k)\,e^{-ikX}, & X \to +\infty, \end{cases} \tag{4.2}$$

where $b(k)$ and $a(k)$ are the coefficients of reflection and transmission respectively.

In [3] (see Section 2.4) the original method for finding the wave-functions ψ and eigenvalues for the reflectionless potential $Q_m = -m(m+1)\operatorname{sech}^2 X$ was described. The general solution y_m of Eq. (4.1) for the potential Q_m connects with the general solution Y_0 for $Q_0 = 0$ by the relationship

$$y_m(X,k) = \prod_{m'=1}^{m} \left(m' \tanh X - \frac{d}{dX} \right) Y_0(X,k), \tag{4.3}$$

and then

$$a(k) = \prod_{m'=1}^{m} \frac{ik+m'}{ik-m'}, \qquad b(k) = 0. \tag{4.4}$$

In our case ($m = 2$) Eq. (4.1) has two bound states

$$\begin{aligned} -ik_1 \equiv \kappa_1 = 1, \qquad & \psi_1 = \sqrt{\frac{3}{2}} \tanh X \operatorname{sech} X, \\ -ik_2 \equiv \kappa_2 = 2, \qquad & \psi_2 = \frac{\sqrt{3}}{2} \operatorname{sech}^2 X. \end{aligned} \tag{4.5}$$

The wave-functions ψ_i are normalized, i.e. $\int_{-\infty}^{+\infty} |\psi_i|^2 dX = 1$, and this conforms to the requirement used in the IST method.

Here the main difference between the VE and the known integrable nonlinear equations appears. It is connected with the existence of only one bound state for the known equations associated with the isospectral Schrödinger equation, while for the VE two bound states occur. Indeed, for the known integrable equations, the potential corresponding to the one-soliton solution has the following dependence on the space coordinate (see Eq. (4.3.9) in [3]):

$$u(x) = 2\eta^2 \operatorname{sech}^2 \eta x. \tag{4.6}$$

It is easy to see that this is related to the case when $m = 1$ in Eq. (3.7), i.e. there is only the one bound state

$$\begin{aligned} &\psi = \sqrt{\eta/2}\ \mathrm{sech}\ \eta x\,, \\ &\psi \sim c\sqrt{\eta}\exp(-\eta x), \quad c = \sqrt{2}, \quad \text{as} \quad x \to +\infty. \end{aligned} \tag{4.7}$$

5. Reconstruction of the One-soliton Solution for the VE

Keeping in mind that there is an incomplete analogy of our problem to the known integrable equations, we shall try to reconstruct the potential (the solution of the transformed VE) from the scattering data as well as to find afterwards the time-dependence for the scattering data and for the one-soliton solution.

As is well known [3], [15], in order to reconstruct the potential for the Schrödinger equation (4.1), we have to know the scattering data. From the relationships (4.5) we obtain, as $X \to \infty$,

$$\begin{aligned} &\psi_1 \sim c_1 e^{-\kappa_1 X}, \qquad c_1 = \sqrt{6}, \qquad \kappa_1 = 1, \\ &\psi_2 \sim c_2 e^{-\kappa_2 X}, \qquad c_2 = \sqrt{12}, \qquad \kappa_2 = 2. \end{aligned} \tag{5.1}$$

Clearly $\kappa_1 = \frac{1}{2}\kappa_2 = 1$ is in agreement with (3.5), (3.6) and (3.4). We shall abandon this condition, i.e. $v = 4$ in Eq. (3.5), and in the final formulas.

For convenience we reproduce the well known procedure for the reconstruction of the potential. The function $B(X;T)$ is constructed from the scattering data (T is the parameter)

$$B(X;T) = \sum_{m=1}^{n} c_m^2(t) e^{-\kappa_m X} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,T) e^{ikX}\,dk.$$

In the next step the following Marchenko–Gelfand–Levitan equation is to be solved [16] for the unknown $K(X,y;T)$:

$$K(X,y;T) + B(X+y;T) + \int_{X}^{\infty} B(y+z;T)K(X,z;T)\,dz = 0. \tag{5.2}$$

The potential is then obtained by means of the relationship

$$-U = Q = -2\frac{d}{dX}K(X,X;T). \tag{5.3}$$

In particular, for the reflectionless potential (3.7), $b(k) = 0$ in (4.2), and the solution can be found in the form

$$K(X,y;T) = -\sum_{m=1}^{N} c_m(T)\psi_m(X;T)e^{-\kappa_m y}. \tag{5.4}$$

This procedure, as is well known, leads to the equation system in ψ_m

$$\mathbf{A}\mathbf{\Psi} = \mathbf{C}, \tag{5.5}$$

where the matrix $\mathbf{A} = [a_{mn}]$ has elements

$$a_{mn} = \delta_{mn} + c_n(T)c_m(T)\frac{e^{-X(\kappa_m+\kappa_n)}}{\kappa_m + \kappa_n},$$

and $\mathbf{\Psi} = [\psi_m]$ and $\mathbf{C} = [c_m(T)e^{-\kappa_m X}]$ are column-vectors.

In Eqs. (5.2)–(5.5) T is a parameter. Although we took $T = 0$ earlier, we preserve the variable T in these relationships in order to use them later to find the time-dependence of the scattering data.

It is known [3], [15] that for a reflectionless potential the value of the determinant $\Delta = \det[a_{mn}]$ is sufficient for reconstructing the potential. Then Eq. (5.3) is reduced to

$$K(X,X;T) = \frac{d\ln|\Delta|}{dX}, \qquad -U = -2\frac{d^2\ln|\Delta|}{dX^2}. \tag{5.6}$$

We use (5.5) and (5.6) to obtain the one-soliton solution of the VE. The scattering data (5.1) and $b(k) = 0$ enable us to find the determinant

$$\Delta = \begin{vmatrix} 1+\frac{c_1^2}{2}e^{-2X} & \frac{c_1c_2}{3}e^{-3X} \\ \frac{c_1c_2}{3}e^{-3X} & 1+\frac{c_2^2}{4}e^{-4X} \end{vmatrix} = (1+e^{-2X})^3 \tag{5.7}$$

and then the potential

$$-U = 12\frac{d}{dX}\frac{e^{-2X}}{1+e^{-2X}} = -6\,\text{sech}^2 X.$$

Thus, we have repeated the standard method for reproducing the potential by means of scattering data (as yet without time-dependence). It is clear from $U = W_X$ and (5.6) that

$$W = 2K(X, X; T).$$

It is noted that the determinant for the one-soliton solution of the KdV equation (3.1) has the form

$$\Delta = 1 + e^{-2x}, \qquad u = 2\operatorname{sech}^2 x. \tag{5.8}$$

The interpretation of (5.7) is important. In the matrix, two states with $q_1 = e^{-X}$ and $q_2 = e^{-2X}$ are involved. Clearly, the time-dependence for an individual state is its own characteristic. However, since these two states relate to the common soliton, there must be a connection between them, i.e. $c_1(T)$ and $c_2(T)$ must be connected. The relation (5.7) determines this connection.

At the beginning we considered the dependence of the potential on the space coordinate, and the time was a parameter. Let us now find the time-dependence of the scattering data $c_1(T)$, $c_2(T)$ that enables us to find the functional dependence of the potential (2.7) on T, i.e. the time-dependence of the one-soliton solution. We start from the relation (see Eq. (22), Chap. 1, Section 2 in [15])

$$\psi(X, k; T) = e^{-ikX} + \int_X^\infty K(X, y; T) e^{-iky} dy. \tag{5.9}$$

Hence, there is a linear operator that reduces the solution e^{-ikX} of the Schrödinger equation with null potential $Q = 0$ to the solution of this equation with the potential $U(X)$. The function $K(X, y; T)$ is the kernel of the transformation operator.

We write Eq. (5.9) for $k = 0$; this procedure is correct and an appropriate theorem has been proved (see Section 3.3 in [3]):

$$\psi(X, k = 0; T) = 1 + \int_X^\infty K(X, y; T) dy. \tag{5.10}$$

Clearly, $\psi(X, k = 0; T) = \varphi(X, T)$, where $\varphi(X, T)$ satisfies the equation system (2.3), (2.4). Taking into account (5.10), we rewrite the relationship (2.3) as

$$1 + \int_X^\infty K(X, y; T)dy = 2\frac{\partial K(X, X, T)}{\partial T} + C. \tag{5.11}$$

Since this equation must be valid at arbitrary X, and taking into account that the function $K(X, y; T) \to 0$ at $|X| \to \infty$, we define the constant of integration $C = 1$. We write, once again, $K(X, y; T)$ as (5.4), because the potential is reflectionless, and we obtain from (5.11)

$$\sum_{m=1}^{2} \frac{c_m(T)}{\kappa_m}\psi_m(X; T)e^{-\kappa_m X} = 2\sum_{m=1}^{2} \frac{\partial c_m(T)\psi_m(X; T)}{\partial T}e^{-\kappa_m X}. \tag{5.12}$$

In this equation we must substitute the values ψ_m that are the solution of system (5.5). Here we consider the values c_m already as functions of T, i.e. $c_m = c_m(T)$. For example ψ_1 is given by

$$\psi_1 = \Delta^{-1}\left(c_1 e^{-\kappa_1 X} + \frac{c_1 c_2^2}{2\kappa_2}e^{-(\kappa_1+2\kappa_2)X} - \frac{c_1 c_2^2}{\kappa_1+\kappa_2}e^{-(\kappa_1+2\kappa_2)X}\right). \tag{5.13}$$

Here Δ is the determinant (5.7) with time-dependence of $c_m = c_m(T)$. We can calculate the following terms which are required for (5.12) (with $\kappa_1 = 1,\ \kappa_2 = 2$):

$$\begin{aligned} &\sum_{m=1}^{2} \frac{c_m(T)}{\kappa_m}\psi_m(X; T)e^{-\kappa_m X} = \Delta^{-1}\left(c_1^2 e^{-2X} + \frac{1}{2}c_2^2 e^{-4X}\right), \\ &\sum_{m=1}^{2} c_m(T)\psi_m(X; T)e^{-\kappa_m X} \\ &\qquad = \Delta^{-1}\left(c_1^2 e^{-2X} + c_2^2 e^{-4X} + \frac{1}{12}c_1^2 c_2^2 e^{-6X}\right). \end{aligned} \tag{5.14}$$

Then, substituting (5.14) into (5.12) and equating to zero the coefficients of e^{-2jX}, $(j = 1, \ldots, 6)$, we obtain the system of differential equations for $c_m(T)$, $(m = 1,\ 2)$

$$
\begin{aligned}
e^{-2X}: &\quad (c_1^2)' = \frac{1}{2}c_1^2,\\
e^{-4X}: &\quad (c_2^2)' = \frac{1}{4}(c_2^2 + c_1^4),\\
e^{-6X}: &\quad \frac{1}{3}(c_1^2c_2^2)' + c_1^2\,(c_2^2)' - c_2^2\,(c_1^2)' = c_1^2c_2^2,\\
e^{-8X}: &\quad c_1^2\,(c_1^2c_2^2)' - c_1^2c_2^2\,(c_1^2)' = \frac{1}{4}\left(c_1^4c_2^2 + 9c_2^4\right),\\
e^{-10X}: &\quad c_2^2\,(c_1^2c_2^2)' - c_1^2c_2^2\,(c_2^2)' = \frac{1}{2}c_1^2c_2^4,\\
e^{-12X}: &\quad c_1^2c_2^2\,(c_1^2c_2^2)' = c_1^2c_2^2\,(c_1^2c_2^2)',
\end{aligned}
\tag{5.15}
$$

where the prime denotes the derivative with respect to time T.

The equation system (5.15) is overdetermined; only the first two equations are independent. Consequently, we solve them with initial conditions $c_1^2(0) = 6,\ c_2^2(0) = 12$. At first, we write the general solution of the system (5.15) as

$$
c_1^2(T) = r_1 e^{T/2}, \qquad c_2^2(T) = r_2 e^{T/4} + \frac{1}{3}r_1^2 e^T, \tag{5.16}
$$

where $r_1,\ r_2$ are arbitrary constants. Hence, in the general case, the time-dependence of the first and second states are different. Nevertheless, we have $r_2 \equiv 0$ due to the relationship between $c_1(0)$ and $c_2(0)$, and then

$$
c_1^2(T) = c_1^2(0)e^{T/2} = 6e^{T/2}, \qquad c_2^2(T) = \frac{1}{3}c_1^4(0)e^T = 12e^T. \tag{5.17}
$$

Thus, the time-dependences satisfy the condition $c_1^2(T)/c_2(T) = \text{const}$. Indeed, if the time-dependence is as in (5.17), the determinant (5.7) can be rewritten as a perfect cube, namely

$$
\Delta = \left(1 + e^{-2(X - T/4)}\right)^3. \tag{5.18}
$$

For convenience, up to this point we have used $\kappa_1 = 1,\ \kappa_2 = 2$. Now we return to one arbitrary parameter κ_1 (with $\kappa_2 = 2\kappa_1$) and rename it as $\alpha \equiv \kappa_1$, and then we obtain

$$
\Delta = \left\{1 + \exp\left[-2\alpha\left(X - \frac{T}{4\alpha^2}\right)\right]\right\}^3. \tag{5.19}
$$

The potential for the one-soliton solution can easily be found by Eq. (5.6)

$$U = 2\frac{d^2 \ln|\Delta|}{dX^2} = 6\alpha^2 \operatorname{sech}^2 \Theta, \qquad \Theta = \alpha\left(X - X_0 - \frac{T}{4\alpha^2}\right). \tag{5.20}$$

For reference we have written the complete equations for finding the solution of Eq. (1.1) in terms of the original variables x, t (for convenience we rename T as $\mu \equiv T$ because here μ is parameter)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0,$$

$$u = \left(\frac{\partial W}{\partial t}\right)_\mu, \quad x = x_0 + \mu + W, \quad W = 2\left(\frac{\partial \ln|\Delta|}{\partial t}\right)_\mu, \tag{5.21}$$

$$\Delta = (1+q^2)^3, \quad q = \exp(-\Theta), \quad \Theta = \alpha\left(t - \frac{\mu - \mu_0}{4\alpha^2}\right), \tag{5.22}$$

$$\alpha = \text{const}, \quad \mu_0 = \text{const}.$$

Thus, we have obtained the one-soliton solution of the VE by using elements of the IST method for the KdV equation. The proposed method is also applicable for finding the two-soliton solution. It is likely that this procedure will shed light upon the formulation of the IST problem that enables one to make progress in the study of the Cauchy problem for the VE (1.1).

6. Two-soliton Solution

In this section we will obtain the two-soliton solution for the VE (1.1). The key for this solution is the value which is assigned to the determinant (5.21) in the one-soliton solution. For more information we rewrite the values (5.22) once again

$$\Delta = (1+q^2)^3, \quad q = \exp\left[-\alpha\left(t - \frac{\mu - \mu_0}{4\alpha^2}\right)\right]. \tag{6.1}$$

It can be seen that there is some analogy to the one-soliton solution of the KdV equation (5.8), namely

$$\Delta = 1 + q^2, \quad q = \exp(\alpha x - 4\alpha^3 t).$$

Moreover, as we noted, the potentials corresponding to the one-soliton solution

(a) for the transformed VE ($T = 0,\ \alpha = 1$)

$$U = 6 \operatorname{sech}^2 X, \tag{6.2}$$

(b) for the KdV equation ($t = 0,\ \eta = 1$)

$$u = 2 \operatorname{sech}^2 x, \tag{6.3}$$

differ from each other by their coefficients. Bearing in mind (5.3) and that $K = \dfrac{\partial \ln|\Delta|}{\partial X}$ (see (5.6)), one can see that the coefficient 6 in (6.2), in contrast to the coefficient 2 in (6.3), is generated by the exponent 3 in relationship (6.1).

Now, if it is recalled that the two-soliton solution for the KdV has the form [11]

$$\begin{gathered}\tilde{F} = \Delta = 1 + q_1^2 + q_2^2 + \tilde{A}_{12} q_1^2 q_2^2, \\ \tilde{A}_{12} = \frac{(\alpha_1 - \alpha_2)^2}{(\alpha_1 + \alpha_2)^2}, \\ q_i = \exp[\alpha_i (x - x_{0i}) - 4\alpha_i^3 t],\end{gathered} \tag{6.4}$$

we can expect that the two-soliton solution for VE can be found in the form (5.21) with F instead of Δ in relation (5.6), where

$$F = \left(1 + q_1^2 + q_2^2 + A_{12} q_1^2 q_2^2\right)^3, \quad q_i = \exp\left[-\alpha_i \left(t - \frac{\mu - \mu_i}{4\alpha_i^2}\right)\right]. \tag{6.5}$$

The value A_{12} is to be found. It should be noted that F is not equal to the determinant Δ of the matrix in (5.5) which is constructed from four states with q_1, q_1^2, q_2, q_2^2, (each soliton has two bound states (4.5))

$$\Delta =$$

$$\begin{vmatrix} 1+3q_1^2 & 2\sqrt{2}q_1^3 & \dfrac{6\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}q_1q_2 & \dfrac{6\sqrt{2\alpha_1\alpha_2}}{\alpha_1+2\alpha_2}q_1q_2^2 \\ 2\sqrt{2}q_1^3 & 1+3q_1^4 & \dfrac{6\sqrt{2\alpha_1\alpha_2}}{2\alpha_1+\alpha_2}q_1^2q_2 & \dfrac{6\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}q_1^2q_2^2 \\ \dfrac{6\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}q_1q_2 & \dfrac{6\sqrt{2\alpha_1\alpha_2}}{2\alpha_1+\alpha_2}q_1^2q_2 & 1+3q_2^2 & 2\sqrt{2}q_2^3 \\ \dfrac{6\sqrt{2\alpha_1\alpha_2}}{\alpha_1+2\alpha_2}q_1q_2^2 & \dfrac{6\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}q_1^2q_2^2 & 2\sqrt{2}q_2^3 & 1+3q_2^4 \end{vmatrix}. \tag{6.6}$$

If the relation $F = \Delta$ were true, we would have $A_{12} = \tilde{A}_{12}$. Moreover, these conditions would lead us to the statement that the problem for scattering data for the VE (1.1) should connect with the isospectral Schrödinger equation. This statement was made in a paper by Hirota and Satsuma [17] as well as in the monograph by Newell, Chapters 3 and 4 [18]. However, because $F \neq \Delta$ and $A_{12} \neq \tilde{A}_{12}$, we can state that the equation system for the IST problem associated with the transformed VE (1.1) does not contain the isospectral Schrödinger equation.

The value A_{12} for (6.5) can be found in the following way. The functional relation (6.5), with A_{12} regarded as unknown, is substituted into Eq. (5.21), and then into Eqs. (2.3) and (2.4). As a consequence of this, a complicated system of equations arises. Without going into details, we note that the procedure is similar to the procedure for finding the system (5.5). Equating to zero the coefficients of e^{jX}, we obtain a system of equations. It turns out that the equations are dependent. As a result we obtain

$$A_{12} = \frac{(\alpha_1-\alpha_2)^2}{(\alpha_1+\alpha_2)^2} \cdot \frac{\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2}{\alpha_1^2+\alpha_2^2+\alpha_1\alpha_2}. \tag{6.7}$$

Thus the relationships (5.21), (6.5), (6.7) are the exact two-soliton solution of the VE (1.1):

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0,$$

$$u = \left(\frac{\partial W}{\partial t}\right)_\mu, \quad x = x_0 + \mu + W, \quad W = 2\left(\frac{\partial \ln|F|}{\partial t}\right)_\mu, \tag{6.8}$$

$$F = \left(1 + q_1^2 + q_2^2 + A_{12}q_1^2q_2^2\right)^3, \quad q_i = \exp(-\Theta_i), \quad \Theta_i = \alpha_i t - \frac{\mu - \mu_i}{4\alpha_i}. \tag{6.9}$$

$$\alpha_i = \text{const}, \quad \mu_i = \text{const}.$$

A similar result has been obtained, independently of the method presented here, in [5] by the means of the Hirota method [11], [12] in some other variables.

7. Interaction of Two Solitons

In the interaction of two solitons for the VE there are features that are not characteristic of the KdV equation. Let us consider the interaction of the two solitons for different ratios α_2/α_1 (see Figs. 1–3). We consider for definiteness that the larger soliton moving with larger velocity catches up with the smaller soliton moving in the same direction. After the nonlinear interaction the solitons separate, their forms are restored but phase-shifts arise. For convenience in the figures the interactions of solitons are shown in coordinates moving with speed $v = 2(\alpha_1^2 + \alpha_2^2)$.

Now we analyze the phase-shift of each soliton. Two moments of time are considered: (a) t_1 when the smaller soliton is far ahead and the larger soliton is at a point $x = 0$ ($\alpha_1 > \alpha_2$, $\mu_1 = 0$, $\mu_2 = \text{const}$, $\mu_2 \gg \alpha_2$); (b) $t_2 \gg t_1$ when the larger soliton leaves the smaller soliton behind.

1. Time $t = t_1 = 0$:

   (a) soliton $u_1 = 6\alpha_1^2 \operatorname{sech}^2(-\Theta_1)$ has its maximum amplitude at $q_1^2 = 1$, $q_2^2 \ll 1$, i.e. at a point $x_{1\,max}(t_1) = x_0 - 6\alpha_1$;

   (b) soliton $u_2 = 6\alpha_2^2 \operatorname{sech}^2(-\Theta_2 + \frac{1}{2}\ln A_{12})$ at $q_1^2 \gg 1$ and $q_2^2 \approx 1$, ($\mu \approx \mu_2$) has its maximum amplitude at a point $x_{2\,max}(t_1) = x_0 + \mu_2 - 2\alpha_2 \ln A_{12} - 6(2\alpha_1 + \alpha_2)$.

2. Time $t = t_2$, ($(\alpha_1^2 - \alpha_2^2)t_2 \gg \mu_2$). Similar analysis shows that the locations where there are maximum soliton amplitudes are:

(a) $x_{1\,max}(t_2) = x_0 - 2\alpha_1 \ln A_{12} + 4\alpha_1^2 t_2 - 6(2\alpha_2 + \alpha_1)$ for soliton u_1 when $q_2^2 \gg 1$, $q_1^2 \approx 1$;

(b) $x_{2\,max}(t_2) = x_0 + \mu_2 + 4\alpha_2^2 t_2 - 6\alpha_2$ for soliton u_2.

Consequently, the larger and smaller solitons have the phase-shifts, respectively

$$\begin{aligned}\delta_1 &= x_{1\,max}(t_2) - x_{1\,max}(t_1) - 4\alpha_1^2 t_2 = -2\alpha_1 \ln A_{12} - 12\alpha_2,\\ \delta_2 &= x_{2\,max}(t_2) - x_{2\,max}(t_1) - 4\alpha_2^2 t_2 = 2\alpha_2 \ln A_{12} + 12\alpha_1.\end{aligned} \tag{7.1}$$

The larger soliton always has a forward phase-shift, i.e. $\delta_1 > 0$, while the smaller soliton can have three kinds of phase-shift. Note that this property is not characteristic of the KdV equation. The different kinds of phase-shift are illustrated in Figs. 1–3. There is a special value of the ratio $(\alpha_2/\alpha_1)^* = 0.88867$.

1. For $\alpha_2/\alpha_1 > (\alpha_2/\alpha_1)^*$ the phase-shift of soliton u_2 is in the opposite direction to the phase-shift of the larger soliton (Fig. 1).

2. For $\alpha_2/\alpha_1 = (\alpha_2/\alpha_1)^*$ the soliton u_2 has no phase-shift (Fig. 2).

3. For $\alpha_2/\alpha_1 < (\alpha_2/\alpha_1)^*$ both solitons have phase-shifts in the same direction (Fig. 3).

The question now arises as to whether the conservation law of the momentum is valid when both solitons have phase-shifts in the same direction. It turns out that this law is not violated at arbitrary phase-shift. This is connected with the following property. The soliton mass determined by the integral $I_1 = \int_{-\infty}^{\infty} u\,dx$ is identically equal to zero. Indeed,

$$I_1 = \int\limits_{-\infty}^{\infty} u\,dx = -\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} = 0.$$

Thus, the arbitrary phase-shift can exist and consequently this does not violate the conservation law of the momentum.

8. Conclusion

The main result of this paper is that we have obtained a way of applying the IST method to the transformed VE. Keeping in mind that the IST is the most appropriate way of tackling the initial value problem, one has to formulate the associated eigenvalue problem. We have proved that the equation system for the IST problem associated with the VE does not contain the isospectral Schrödinger equation. Nevertheless, the analysis of the transformed VE in the context of the isospectral Schrödinger equation allowed us to obtain the two-soliton solution.

Once this investigation was completed, we made some progress in the formulation of the IST for the transformed VE; we found that the spectral problem associated with the VE is of third order [14].

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$\alpha_2/\alpha_1 = 0.99$

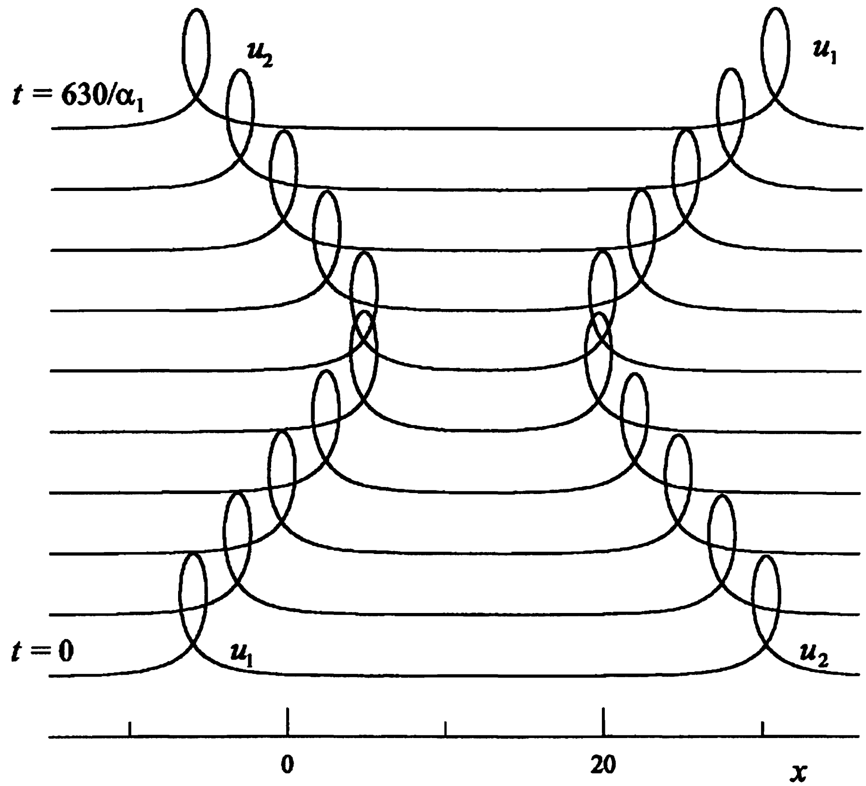

Figure 1: Interaction of two solutions in moving coordinates at time interval $\Delta t = 70/\alpha_1$

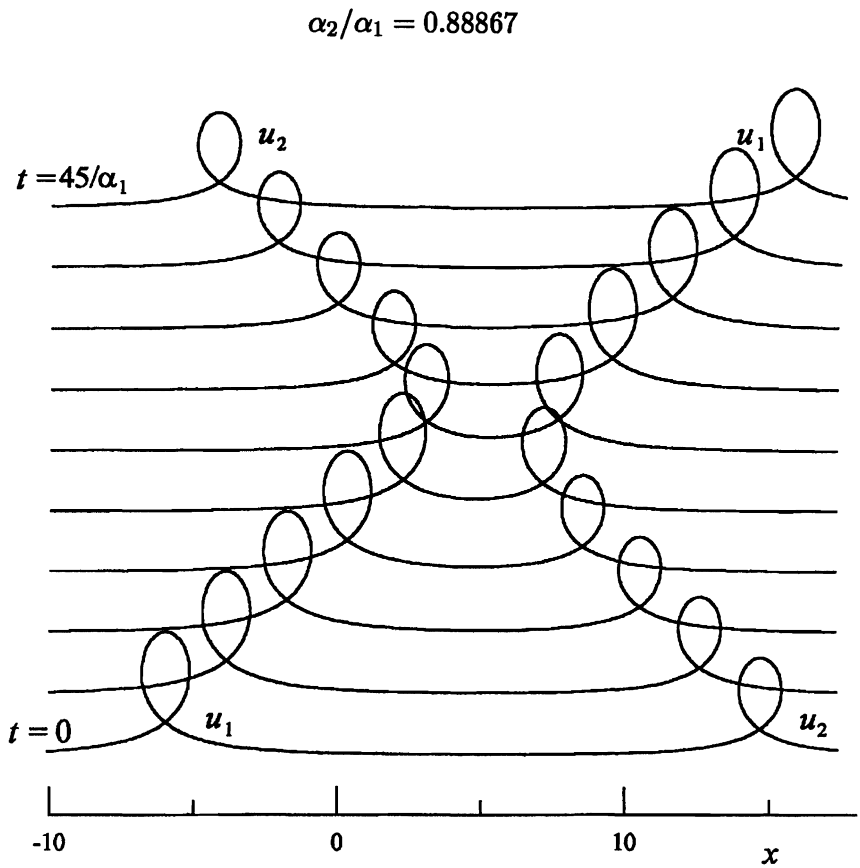

Figure 2: The phase-shift of the smaller soliton is zero.
The time interval is $\Delta t = 5/\alpha_1$

$\alpha_2/\alpha_1 = 0.5$

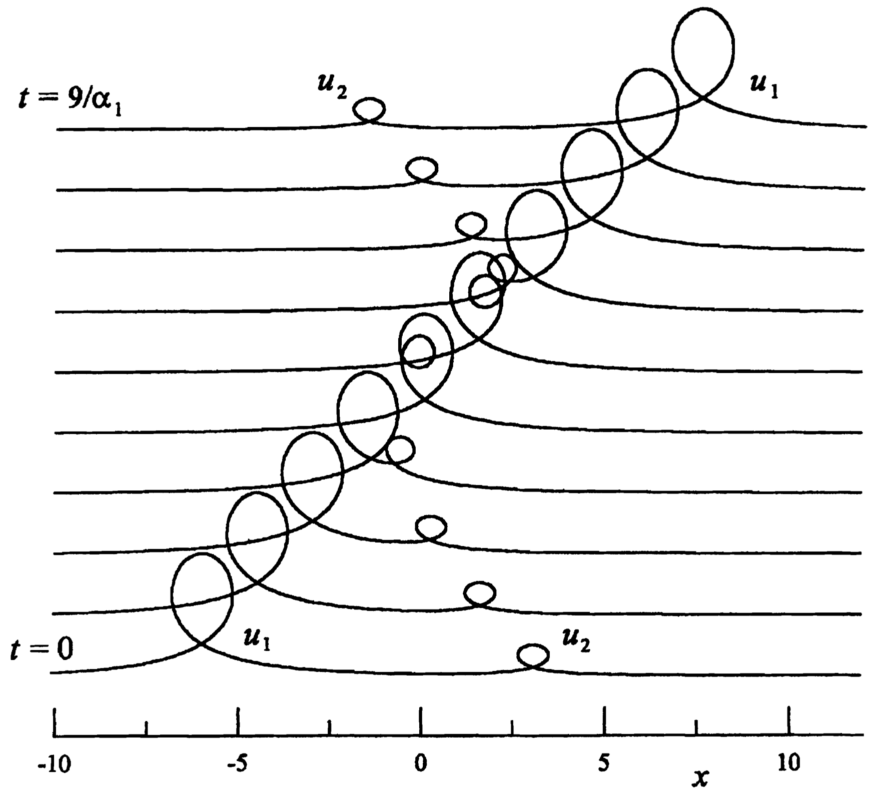

Figure 3: Both solitons have phase-shifts in the same direction. The time interval is $\Delta t = 1/\alpha_1$