UDC 539

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Integrability of a novel nonlinear evolution equation

(Presented by Corresponding Member of the NAS of Ukraine A. G. Zagorodnii)

Для нового узагальненого еволюційного рівняння виведено перетворення Беклунда. За допомогою цього перетворення доведено, що існує нескінченний ряд величин, що зберігаються. Знайдена пара Лакса; вона утримує спектральне рівняння третього порядку. Методом оберненої задачі розсіювання відтворюється точний N-солітонний розв'язок. Результат ілюструється одно- та двосолітонним розв'язками.

1. We consider the generalized equation which was first suggested in [1], namely

$$\frac{\partial}{\partial x}\left(\mathcal{D}^2 u + \tfrac{1}{2}u^2 + \beta u\right) + \mathcal{D}u = 0 \qquad \text{or} \qquad \left(\frac{\partial u}{\partial x} + \mathcal{D}\right)\left(\frac{\partial}{\partial x}\mathcal{D}u + u + \beta\right) = 0, \tag{1.1}$$

where $\mathcal{D} := \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x}$ and β is a real arbitrary constant. This equation is a generalization of the Vakhnenko equation (VE) that we have investigated recently (see [2, 3] and references therein).

In [1], we found the N-soliton solution to Eq. (1.1) via the use of Hirota's method. The key step in finding the solutions is to introduce the transformations

$$x = \theta(X,T) := T + W(X,T) + x_0, \qquad t = X, \qquad W = \int_{-\infty}^{X} U(X',T)\,dX', \tag{1.2}$$

where x_0 is a constant, $u(x,t) = U(X,T) = W_X(X,T)$, and it is assumed that, as $|X| \to \infty$, the derivatives of W vanish and W tends to a constant. In terms of the new variables, Eq. (1.1) may be written as

$$U_{XXT} + UU_T + U_X \int_{-\infty}^{X} U_T(X',T)\,dX' + U_X + \beta U_T = 0, \tag{1.3}$$

or equivalently

$$W_{XXT} + (1 + W_T)W_X + \beta W_T = 0. \tag{1.4}$$

This is the transformed version of Eq. (1.1). The corresponding bilinear equation is

$$(D_X^3 D_T + D_X^2 + \beta D_X D_T) f \cdot f = 0, \tag{1.5}$$

where D is the Hirota D operator [4] and $W = 6\left(\ln f(X,T)\right)_X$.

With $\beta = 0$, (1.4) and (1.5) reduce to Eqs. (2.6) and (2.9) in [2], respectively. With $\beta = -1$ and $T \to -T$, (1.4) and (1.5) are associated with the Hirota – Satsuma equation (HSE) for shallow water waves [4, 5]. The solution to the HSE by Hirota's method is given in [5]; however, as far as we are aware, the solution by the inverse scattering transform (IST) method has not been given explicitly in the literature.

The main aim of the present paper is to extend the investigation of Eq. (3) in [3] by the IST method to Eq. (1.1).

2. Bäcklund transformation and Lax pair. We follow the method developed in [6] and show that the Bäcklund transformation for (1.5) is given by the equations

$$(D_X^3 + \beta D_X - \lambda(X))f' \cdot f = 0, \tag{2.1}$$

$$(3D_X D_T + 1 + \mu(T)D_X)f' \cdot f = 0, \tag{2.2}$$

where $\lambda(X)$ is an arbitrary function of X and $\mu(T)$ is an arbitrary function of T.

Consider the expression P defined by

$$P := [(D_T D_X^3 + D_X^2 + \beta D_X D_T)f' \cdot f']ff - f'f'[(D_T D_X^3 + D_X^2 + \beta D_X D_T)f \cdot f], \tag{2.3}$$

where $f \neq f'$. In [2], it was shown that

$$[(D_T D_X^3 + D_X^2)f' \cdot f']ff - f'f'[(D_T D_X^3 + D_X^2)f \cdot f] =$$
$$= 2D_T(D_X^3 f' \cdot f) \cdot (f'f) - 2D_X(\{3D_T D_X + 1\}f' \cdot f) \cdot (D_X f' \cdot f). \tag{2.4}$$

By using (2.4) and identities (II.1) and (VII.2) from [7], P given by (2.3) can be reduced to the form

$$P = 2D_T(\{D_X^3 + \beta D_X - \lambda(X)\}f' \cdot f) \cdot (f'f) -$$
$$- 2D_X(\{3D_T D_X + 1 + \mu(T)D_X\}f' \cdot f) \cdot (D_X f' \cdot f). \tag{2.5}$$

It is clear from (2.5) that if (2.1) and (2.2) hold, then $P = 0$. Furthermore, it then follows from (2.3) that if f is a solution of (1.5), then so is f' and vice versa. Consequently, we have proved that Eqs. (2.1) and (2.2) constitute a Bäcklund transformation for (1.5). As expected, with $\beta = -1$ and $T \to -T$, (2.1) and (2.2) become the Bäcklund transformation for the HSE (see (5.131) and (5.132) in [4]).

Now we show that the IST problem for Eq. (1.4) has a third-order eigenvalue problem that is similar to the one associated with a higher order KdV equation [8, 9], a Boussinesq equation [8, 10, 11], and the HSE [4, 12].

Introducing the function

$$\psi = f'/f \tag{2.6}$$

and taking into account (1.4), we find that (2.1) and (2.2) reduce to

$$\psi_{XXX} + (\beta + W_X)\psi_X - \lambda\psi = 0, \tag{2.7}$$

$$3\psi_{XT} + (1 + W_T)\psi + \mu\psi_X = 0, \tag{2.8}$$

respectively, where we have used results similar to (X.1)–(X.3) in [4].

Computing the cross-derivative condition $(\psi_{XXX})_T = (\psi_{XT})_{XX}$ from (2.7) and (2.8) and using (2.7) and (2.8) again in order to eliminate any derivative of ψ higher than $(3,0)$ or $(1,1)$ in (X,T), we obtain the following equation linear in ψ_T, ψ_X, ψ_{XX}, and ψ:

$$3\lambda\psi_T + (1 + W_T)\psi_{XX} - W_{XT}\psi_X +$$
$$+ [W_{XXT} + (\beta + W_X)(1 + W_T) + \mu(T)\lambda(X)]\psi = 0. \tag{2.9}$$

The integrability condition of system (2.7) and (2.8), or (2.7) and (2.9), is

$$[W_{XXT} + (1 + W_T)W_X + \beta W_T]_X\psi + \lambda_X(3\psi_T + \mu\psi) = 0.$$

Hence, if (1.4) holds, $\lambda_X = 0$ and so the spectrum λ of (2.7) remains constant. The constant λ is what is required in the IST problem. Therefore, we obtain the following third-order Lax pair for Eq. (1.4):

$$\psi_{XXX} + (\beta + W_X)\psi_X - \lambda\psi = 0, \tag{2.10}$$

$$3\lambda\psi_T + (1 + W_T)\psi_{XX} - W_{XT}\psi_X + (\beta + \lambda\mu(T))\psi = 0. \tag{2.11}$$

Following the procedure given in [4, 9], we can rewrite (2.10) and (2.11) in terms of the potential W. Recalling that $\psi = f'/f$ and noting that $W' - W = 6\varphi_X$ and $W' + W = 6\rho_X$, where $\varphi = \ln f'/f$ and $\rho = \ln f'f$, we find that (2.10) and (2.11) give the following Bäcklund transformation in ordinary form:

$$(W' - W)_{XX} + \tfrac{1}{2}(W' - W)(W' + W)_X + \tfrac{1}{36}(W' - W)^3 + \beta(W' - W) - 6\lambda = 0, \tag{2.12}$$

$$3\lambda(W' - W)_T + \left[(1 + W_T)((W' - W)_X + \tfrac{1}{6}(W' - W)^2) - W_{XT}(W' - W)\right]_X = 0. \tag{2.13}$$

A systematic way to derive higher conservation laws via the Bäcklund transformation has been developed by Satsuma; he applied it to the KdV equation [13]. Later Satsuma and Kaup [9] applied the method to a higher order KdV equation. Since, with constant λ, our Eq. (2.12) is (apart from a scaling factor) the same as Eq. (29) in [9], and our Eq. (2.13) is in conservation form, we can apply the results on higher conservation laws in §4 of [9] to Eq. (1.4). Thus, we deduce that Eq. (1.4) has an infinite sequence of conservation laws. For example, the first two nontrivial conserved densities are U and $(U^3 - 3U_X^2 + 3\beta U^2)$.

3. The N-soliton solution. The inverse problem for certain third-order spectral equations has been considered by Kaup [8] and Caudrey [10, 14]. As expected, with $\beta = -1$ and $T \to -T$, (2.7) and (2.8) are the corresponding equations for the HSE (see Eqs. (A8a) and (A8b) in [12]). In [15], it is noted that the scattering problem for the HSE is similar to that for the Boussinesq equation [11].

The general theory of the inverse scattering problem for N spectral equations has been developed in [10]. Following [10], the spectral equation (2.10) can be rewritten as

$$\frac{\partial}{\partial X}\boldsymbol{\psi} = [\mathbf{A}(\zeta) + \mathbf{B}(X, \zeta)] \cdot \boldsymbol{\psi}, \tag{3.1}$$

$$\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \psi_X \\ \psi_{XX} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -\beta & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -W_X & 0 \end{pmatrix}.$$

We find the eigenvalue $\lambda_j(\zeta)$ of the matrix $\mathbf{A}$ from the equation $\det(\mathbf{A} - \lambda_j\mathbf{E}) = -\lambda_j^3 - \beta\lambda_j + \lambda = 0$ ($\mathbf{E}$ is the identity matrix). This relation between the values λ and λ_j can be rewritten in parametric form with ζ as parameter, namely

$$\lambda_j = (\beta/3)^{1/2}(\omega_j\zeta - (\omega_j\zeta)^{-1}), \qquad \lambda = (\beta/3)^{3/2}(\zeta^3 - \zeta^{-3}), \tag{3.2}$$

where $\omega_j = e^{i2\pi(j-1)/3}$ are the cubes of roots of 1 ($j = 1, 2, 3$). Because of the properties $\lambda_1(\zeta) = \lambda_1(-\zeta^{-1})$, $\lambda_2(\zeta) = \lambda_3(-\zeta^{-1})$, $\lambda_3(\zeta) = \lambda_2(-\zeta^{-1})$, and $\lambda(\zeta) = \lambda(-\zeta^{-1})$, it is sufficient to consider the values ζ located outside (or inside) of the circle $|\zeta| = 1$ only.

The right- and left-eigenvectors are

$$\boldsymbol{v}_j(\zeta) = \left(1,\ \lambda_j,\ \lambda_j^2\right)^T, \qquad \widetilde{\boldsymbol{v}}_j(\zeta) = \left(\lambda_j^2 + \beta,\ \lambda_j,\ 1\right). \tag{3.3}$$

It should be noted that the passage to the limit $\beta \to 0$ must be carried out with $\sqrt{\beta}\zeta$ held constant.

The solution of the linear equation (2.10) (or equivalently (3.1)) has been obtained by Caudrey (see system (4.5) in [10]) in terms of Jost functions $\boldsymbol{\phi}_j(X,\zeta)$ which have the asymptotic behaviour

$$\boldsymbol{\Phi}_j(X,\zeta) := \exp\{-\lambda_j(\zeta)X\}\boldsymbol{\phi}_j(X,\zeta) \to \boldsymbol{v}_j(\zeta) \qquad \text{as} \quad X \to -\infty. \tag{3.4}$$

We consider only the N-soliton solution by putting $Q_{ij}(\zeta) \equiv 0$ in (6.20) from [10]. Then there is only the bound state spectrum which is associated with the soliton solutions.

We follow the procedure described in [3]. Let the bound state spectrum be defined by K poles located, for definiteness, outside the circle $|\zeta| = 1$. Relation (6.20) from [10] is reduced to the form

$$\Phi_1(X,\zeta) = 1 - \sum_{k=1}^{K}\sum_{j=2}^{3} \gamma_{1j}^{(k)} \frac{\exp\{[\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(k)})]X\}}{\lambda_1(\zeta_1^{(k)}) - \lambda_1(\zeta)} \Phi_1(X, \omega_j\zeta_1^{(k)}). \tag{3.5}$$

This relation (3.5) involves the spectral data, namely the poles $\zeta_i^{(k)}$ and the quantities $\gamma_{ij}^{(k)}$. As in [3], we need only to consider the function $\Phi_1(X,\zeta)$ since there is a set of symmetry properties. One can prove similarly to [3] that $\operatorname{Re}\lambda = 0$ for compact support and there is the relation between Jost functions:

$$\phi_{1X}(X,\zeta) = \frac{i}{\sqrt{3}}\left[\phi_{1X}(X,-\omega_2\zeta)\phi_1(X,-\omega_3\zeta) - \phi_{1X}(X,-\omega_3\zeta)\phi_1(X,-\omega_2\zeta)\right]. \tag{3.6}$$

As is for Eq. (3) in [3], the poles appear in pairs, wherein the properties for the pair n ($n = 1, 2, \ldots, N$, and $2N = K$) are

$$\text{(i)} \quad \zeta_1^{(2n-1)} = i\omega_2\xi_n, \qquad \text{(ii)} \quad \zeta_1^{(2n)} = -i\omega_3\xi_n. \tag{3.7}$$

Since U is real and λ is imaginary, either ξ_n is real when $\beta > 0$ or ξ_n is imaginary when $\beta < 0$, i.e., $\sqrt{\beta}\xi_n$ is real.

By considering Eq. (3.6) in the vicinity of the first pole $\zeta_1^{(2n-1)}$ of the pair n and using (3.5), one can obtain a relation between $\gamma_{12}^{(2n-1)}$ and $\gamma_{13}^{(2n)}$. Substituting $\phi_1(X,\zeta)$ in the form given by (3.4) and (3.5) into (3.6) and letting $X \to -\infty$, we have $\gamma_{12}^{(2n)} = \gamma_{13}^{(2n-1)} = 0$ and the ratio $\gamma_{12}^{(2n-1)}/\gamma_{13}^{(2n)} = (\omega_2\xi_n + (\omega_2\xi_n)^{-1})/(\omega_3\xi_n + (\omega_3\xi_n)^{-1})$. Therefore, the properties of $\gamma_{ij}^{(k)}$ should be defined by the relationships

$$\left.\begin{aligned} &\text{(i)} \quad \gamma_{12}^{(2n-1)} = \sqrt{\beta}\gamma_n\left(\omega_2\xi_n + (\omega_2\xi_n)^{-1}\right), && \gamma_{13}^{(2n-1)} = 0, \\ &\text{(ii)} \quad \gamma_{12}^{(2n)} = 0, && \gamma_{13}^{(2n)} = \sqrt{\beta}\gamma_n\left(\omega_3\xi_n + (\omega_3\xi_n)^{-1}\right), \end{aligned}\right\} \tag{3.8}$$

where γ_n are arbitrary constants. We will show below that γ_n is real when W_X is real.

Omitting the details, which can be found in [3], we indicate that the T-evolution of the scattering data is given by the relationships (with $k = 1, 2, \ldots, K$)

$$\left.\begin{aligned} &\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0), \\ &\gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0) \exp\left\{\left[-\left(3\lambda_j(\zeta_1^{(k)})\right)^{-1} + \left(3\lambda_1(\zeta_1^{(k)})\right)^{-1}\right]T\right\}, \end{aligned}\right\} \tag{3.9}$$

and then the N-soliton solution of Eq. (1.4) is

$$U(X,T) = W_X(X,T) = 3\frac{\partial^2}{\partial X^2} \ln\left(\det M(X,T)\right), \tag{3.10}$$

where M is the $2N \times 2N$ matrix given by

$$\begin{aligned} M_{kl} = \delta_{kl} - \sum_{j=2}^{3} \gamma_{1j}^{(k)}(0) \times \\ \times \frac{\exp\left\{\left[-\left(3\lambda_j(\zeta_1^{(k)})\right)^{-1} + \left(3\lambda_1(\zeta_1^{(k)})\right)^{-1}\right]T + \left(\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})\right)X\right\}}{\lambda_j(\zeta_1^{(k)}) - \lambda_1(\zeta_1^{(l)})} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &n = 1, 2, \ldots, N, \qquad m = 2n - 1, \\ &\lambda_1(\zeta_1^{(m)}) = \sqrt{\beta/3}\left(i\omega_2\xi_m - (i\omega_2\xi_m)^{-1}\right), \qquad \lambda_2(\zeta_1^{(m)}) = \sqrt{\beta/3}\left(i\omega_3\xi_m - (i\omega_3\xi_m)^{-1}\right), \\ &\gamma_{12}^{(m)}(0) = \sqrt{\beta}\gamma_m(0)\left(\omega_2\xi_m + (\omega_2\xi_m)^{-1}\right), \qquad \gamma_{13}^{(m)} = 0, \\ &\lambda_1(\zeta_1^{(m+1)}) = \sqrt{\beta/3}\left(-i\omega_3\xi_m + (i\omega_3\xi_m)^{-1}\right), \quad \lambda_3(\zeta_1^{(m+1)}) = \sqrt{\beta/3}\left(-i\omega_2\xi_m + (i\omega_2\xi_m)^{-1}\right), \\ &\gamma_{12}^{(m+1)} = 0, \qquad \gamma_{13}^{(m+1)}(0) = \sqrt{\beta}\gamma_m(0)\left(\omega_3\xi_m + (\omega_3\xi_m)^{-1}\right). \end{aligned}$$

For the N-soliton solution, there are N arbitrary constants ξ_m and N arbitrary constants γ_m.

Finally, the N-soliton solution of Eq. (1.1) is given in parametric form by $u(x,t) = U(t,T)$, $x = \theta(t,T)$, where $\theta(X,T)$ is defined in (1.2).

In passing, we note that, in the limit $\beta \to 0$ with $\sqrt{\beta}\xi_i$ held constant, the solution given above reduces to the result for Eq. (1) obtained in [3].

4. Examples of one- and two-soliton solutions. In order to obtain the one-soliton solution of Eq. (1.4), we need first to calculate the 2×2 matrix M according to (3.11) with $N = 1$. The elements of the matrix are

$$\begin{aligned} &M_{11} = 1 - \frac{\sqrt{\beta}\gamma_1}{2k}\left(\omega_2\xi_1 + \frac{1}{\omega_2\xi_1}\right)\exp\left[2k(X - cT)\right], \\ &M_{12} = -\frac{\sqrt{3}\gamma_1}{2}\left\{\left(\omega_2\xi_1 + \frac{1}{\omega_2\xi_1}\right) \bigg/ \left(i\omega_3\xi_1 - \frac{1}{i\omega_3\xi_1}\right)\right\} \times \\ &\qquad \times \exp\left[2\sqrt{\beta/3}\left(i\omega_3\xi_1 - \frac{1}{i\omega_3\xi_1}\right)X - 2kcT\right], \end{aligned}$$

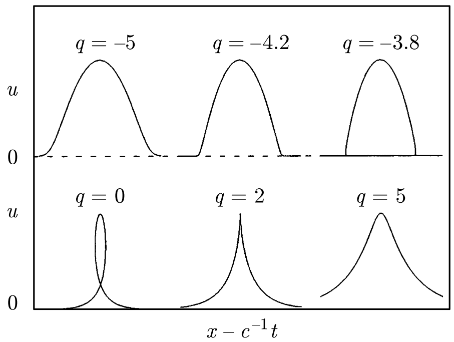

Fig. 1. Hump-like $u(x,t)$ for $q = -5$, -4.2 and 5; loop-like $u(x,t)$ for $q = -3.8$ and 0, and the cusp-shaped $u(x,t)$ for $q = 2$

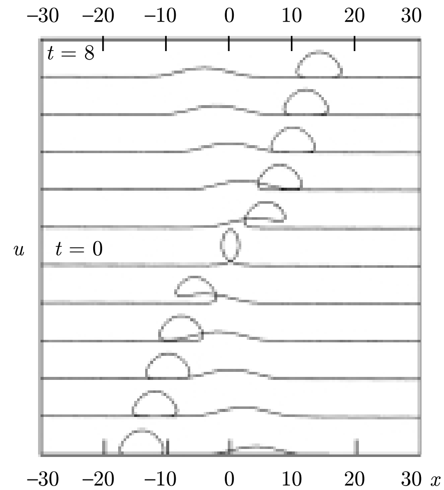

Fig. 2. Interaction process for $r = 0.6$ and $s = -2.7$

$$M_{21} = \frac{\sqrt{3}\gamma_1}{2}\left\{\left(\omega_3\xi_1 + \frac{1}{\omega_3\xi_1}\right)\Big/\left(i\omega_2\xi_1 - \frac{1}{i\omega_2\xi_1}\right)\right\}\times$$
$$\times\exp\left[2\sqrt{\beta/3}\left(-i\omega_2\xi_1 + \frac{1}{i\omega_2\xi_1}\right)X - 2kcT\right],$$
$$M_{22} = 1 - \frac{\sqrt{\beta}\gamma_1}{2k}\left(\omega_3\xi_1 + \frac{1}{\omega_3\xi_1}\right)\exp\left[2k(X - cT)\right],$$

and the determinant of the matrix is

$$\det M = \left\{1 + \frac{\gamma_1}{2}\frac{\xi_1^2 + 1}{\xi_1^2 - 1}\exp\left[2k(X - cT)\right]\right\}^2, \tag{4.1}$$

where $k = \sqrt{\beta}(\xi_1 - \xi_1^{-1})/2$ and $c^{-1} = \beta(\xi_1^2 + \xi_1^{-2} - 1)$. Notice that this determinant is a perfect square.

Thus, from (3.10), the one-soliton solution of Eq (1.4), as obtained by the IST method, is

$$U(X,T) = 6k^2\,\mathrm{sech}^2\left[k(X - cT) + \alpha_1\right], \tag{4.2}$$

where $\alpha_1 = \frac{1}{2}\ln\left[\frac{\gamma_1}{2}\left(\frac{\xi_1^2 + 1}{\xi_1^2 - 1}\right)\right]$ is an arbitrary constant. Since U is real, it follows from (4.2) that α_1 is real; moreover, since $\sqrt{\beta}\xi_1$ is real, γ_1 is also real.

In a similar way (details are omitted), we find that, for the two-soliton solution, M is a 4×4 matrix for which

$$\det M = \left(1 + q_1^2 + q_2^2 + b_{12}^2 q_1^2 q_2^2\right)^2, \qquad q_i = \exp\left[2k_i(X - c_iT) + \alpha_i\right], \tag{4.3}$$

$$b_{12}^2 = \left(\frac{\xi_3 - \xi_3^{-1} - \xi_1 + \xi_1^{-1}}{\xi_3 - \xi_3^{-1} + \xi_1 - \xi_1^{-1}}\right)^3 \frac{\xi_3^3 - \xi_3^{-3} + \xi_1^3 - \xi_1^{-3}}{\xi_3^3 - \xi_3^{-3} - \xi_1^3 + \xi_1^{-3}}, \qquad \alpha_i = \frac{1}{2}\ln\left[\frac{\gamma_{2i-1}}{2}\left(\frac{\xi_{2i-1}^2 + 1}{\xi_{2i-1}^2 - 1}\right)\right],$$

$$k_i = \sqrt{\beta}(\xi_{2i-1} - \xi_{2i-1}^{-1})/2, \qquad c_i^{-1} = \beta(\xi_{2i-1}^2 + \xi_{2i-1}^{-2} - 1).$$

Note that the determinant in (4.3) is a perfect square. Relationship (3.10) together with (4.3) gives the two-soliton solution of (1.4).

Relations (4.2) and (4.3) agree with the one- and two-soliton solutions as found by Hirota's method [1]. A novel feature of Eq. (1.1) is that different types of soliton solutions are possible, namely hump-like, cusp-like, or loop-like. As discussed in detail in [1], the shape of the one-soliton solution to Eq. (1.4) depends on the value of $q := \beta/k^2$. Examples of loop-like solitons, hump-like solitons, and a solution with cusp are illustrated in Fig. 1, where u is plotted against $x - c^{-1}t$. The interaction process between the solitons in the two-soliton solution to Eq. (1.1) was also discussed in detail in [1]. The character of the interaction depends on the values of $r := k_1/k_2$ and $s := \beta/k_2^2$. The example of the interaction is given in Fig. 2, where u is plotted against x at several equally spaced values of t.

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