O.O. VAKHNENKO,[1] V.O. VAKHNENKO[2]

[1] Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14-B, Metrologichna Str., Kyïv 03680, Ukraine; e-mail: vakhnenko@bitp.kiev.ua)

[2] Department of Dynamics of Deformable Solids, Subbotin Institute of Geophysics, Nat. Acad. of Sci. of Ukraine
(63-B, Bohdan Khmel'nyts'kyy Str., Kyïv 01054, Ukraine; e-mail: vakhnenko@ukr.net)

PACS 02.03.Ik, 11.10.Lm, 45.05.+x, 07.50.Ek

LINEAR ANALYSIS OF EXTENDED INTEGRABLE NONLINEAR LADDER NETWORK SYSTEM

The nontrivial integrable extension of a nonlinear ladder electric network system characterized by two coupling parameters is presented. Relying upon the lowest local conservation laws, the concise form of the general semidiscrete integrable system is given, and two versions of its self-consistent reduction in terms of four true field variables are found. The comprehensive analysis of the dispersion equation for low-amplitude excitations of the system is made. The criteria distinguishing the two-branch and four-branch realizations of the dispersion law are formulated. The critical values of adjustable coupling parameter are found, and a collection of qualitatively distinct realizations of the dispersion law is graphically presented. The loop-like structure of the low-amplitude dispersion law of a reduced system emerging within certain windows of the adjustable coupling parameter turns out to reproduce the loop-like structure of the dispersion law typical of beam-plasma oscillations in hydrodynamic plasma. The richness of the low-amplitude spectrum of the proposed ladder network system as a function of the adjustable coupling parameter is expected to stimulate even the more rich dynamical behavior in an essentially nonlinear regime.

Keywords: nonlinear ladder electric network system, dispersion law, hydrodynamic plasma.

1. Introduction

After the discovery of first integrable nonlinear dynamical models on a regular one-dimensional lattice [1–4], the interest in the development of new integrable semidiscrete nonlinear systems has been steadily supported by the wide range of physical problems, where the spatial discreteness and regularity play a crucial role. Among the most typical physical objects, where the semidiscrete nonlinear systems found their applications, are the optical waveguide arrays [5], semiconductor superlattices [6, 7], electric superstructures [8], as well as the regular macromolecular structures of both natural [9] and synthetic [10] origins.

In two recent articles [11, 12], we have proposed early unknown integrable semidiscrete nonlinear multifield systems associated with the new type of fourth-order spectral problems. The number of field variables in a particular system was determined by the adopted reduction and can never be decreased lower than six truly independent field variables. Thus, even in its simplest realization, the analysis of the system appears to be rather complicated.

One of the ways to obtain more simple but still early unknown systems is to reduce the order of an auxiliary spectral problem associated with a tentative system within a zero-curvature scheme. In so doing, it is reasonable to demand some elements of contiguity between the antecendent and sought-for schemes. Otherwise, the procedure of empirical selection of an auxiliary spectral operator consistent with a proper auxiliary evolution operator in the framework of the zero-curvature approach may fail to be fruitful.

The above observation, when combined with the Caudrey definition of the order of a spectral operator [13–15], has allowed us to reveal the constructive version of the early unknown third-order auxiliary spectral operator giving rise to new integrable systems. In this paper, we shall present the real-field realization of a general semidiscrete nonlinear integrable system and carry on the analysis of its low-amplitude excitations related to a reduced system given in terms of symmetric field variables. On the other hand, we shall propose an alternative reduced version of the system allowing to treat it as the nontrivial extension of a ladder-like nonlinear electric transmission line, consisting of nonlinear inductors and nonlinear capacitors and influenced by some vibrational subsystem.

The main aim of the paper was to develop a semidiscrete integrable model suitable to analyze the complex nonlinear dynamical phenomena presumably in ladder-like nonlinear electric networks coupled with the vibrational degrees of freedom. As the first step, we will show that, even in the linear regime, the proposed model demonstrates an essentially nontrivial behavior as a function of the additional adjustable coupling parameter in contrast to the majority of already known integrable models exhibiting rather simple low amplitude dispersion laws due to the lack of an additional coupling.

2. Auxiliary Operators Mutually Consistent in the Framework of Zero-Curvature Equation

In order to ensure the integrability of a desired nonlinear system, one need to approbate the zero-curvature equation [16]

$$\dot{M}(n|z) = B(n+1|z)M(n|z) - M(n|z)B(n|z) \quad (2.1)$$

by the spectral $M(n|z)$ and evolution $B(n|z)$ operators chosen properly among nonsingular matrices. Here, the dot written over the operator $M(n|z)$ on the left-hand side of the zero-curvature equation (2.1) means the differentiation with respect to the time τ, the integer n denotes the discrete spatial coordinate running from minus to plus infinity, while z denotes the auxiliary spectral parameter independent of time.

The arguments given in Introduction prompt us to define the spectral operator $M(n|z)$ as a 3×3 matrix,

$$M(n|z) = \begin{pmatrix} z^2 + T(n) & \beta F_+(n)z + \alpha F_+(n) & G_+(n)z + G_-(n)z^{-1} \\ \alpha F_-(n)z + \beta F_-(n) & 0 & \alpha F_-(n) + \beta F_-(n)z^{-1} \\ G_-(n)z + G_+(n)z^{-1} & \beta F_+(n) + \alpha F_+(n)z^{-1} & T(n) + z^{-2} \end{pmatrix}, \quad (2.2)$$

and to seek the evolution operator $B(n|z)$ in the form

$$B(n|z) = \begin{pmatrix} a(n)z^2 + d(n) & \beta b_+(n)z + \alpha b_+(n) & c_+(n)z + c_-(n)z^{-1} \\ \alpha b_-(n)z + \beta b_-(n) & d(n) - c(n) & \alpha b_-(n) + \beta b_-(n)z^{-1} \\ c_-(n)z + c_+(n)z^{-1} & \beta b_+(n) + \alpha b_+(n)z^{-1} & d(n) + a(n)z^{-2} \end{pmatrix}, \quad (2.3)$$

where α and β are some fitting parameters independent of the time. Then the direct calculations based on the zero-curvature equation (2.1) confirm our conjecture and allow us to decipher almost all matrix elements $B_{jk}(n|z)$ of the tested evolution operator $B(n|z)$ through the matrix elements $M_{jk}(n|z)$ of the chosen spectral operator $M(n|z)$ provided

$$\alpha^2 + \beta^2 = 0. \quad (2.4)$$

Thus, for the functions entering into the evolution operator $B(n|z)$, we have

$$a(n) = k, \quad (2.5)$$

$$b_+(n) = kF_+(n), \quad (2.6)$$

$$b_-(n) = kF_-(n-1), \quad (2.7)$$

$$c_+(n) = kG_+(n), \quad (2.8)$$

$$c_-(n) = kG_-(n-1), \quad (2.9)$$

$$d(n) = -k\alpha\beta F_+(n)F_-(n-1) - kG_+(n)G_-(n-1), \quad (2.10)$$

where k is an arbitrary (time-dependent, in general) parameter. The only exception is the sampling function $c(n)$ which, similarly to other precedents [17, 18], remains arbitrary for the time being.

Here and latter on, we adopt the parameters k and $\alpha\beta$ to be the real-valued ones.

3. General Semidiscrete Nonlinear Evolution Equations

We call the quantities $F_+(n)$, $F_-(n)$, $G_+(n)$, $G_-(n)$, and $T(n)$ to be the prototype field variables. According to the zero-curvature equation (2.1) and expressions (2.2) and (2.3), (2.5)–(2.10) for the spectral $M(n|z)$ and evolution $B(n|z)$ operators, their evolution is described by the following collection of semidiscrete nonlinear equations:

$$\frac{d}{d\tau}\ln F_+(n) = kG_+(n+1) - kG_+(n) - kT(n) - k\alpha\beta F_+(n+1)F_-(n) - kG_+(n+1)G_-(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1) + c(n), \tag{3.1}$$

$$\frac{d}{d\tau}\ln F_-(n) = kG_-(n) - kG_-(n-1) + kT(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1) - k\alpha\beta F_+(n+1)F_-(n) - kG_+(n+1)G_-(n) - c(n+1), \tag{3.2}$$

$$\frac{d}{d\tau}\ln\Big[1 - T(n) + G_+(n) - G_-(n)\Big] = kG_+(n) - kG_+(n+1) + kG_-(n) - kG_-(n-1) - k\alpha\beta F_+(n+1)F_-(n) - kG_+(n+1)G_-(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1), \tag{3.3}$$

$$\frac{d}{d\tau}\ln\Big[1 + T(n) - G_+(n) - G_-(n)\Big] = kG_+(n) - kG_+(n+1) - kG_-(n) + kG_-(n-1) - k\alpha\beta F_+(n+1)F_-(n) - kG_+(n+1)G_-(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1), \tag{3.4}$$

$$\frac{d}{d\tau}\ln\Big[1 + T(n) - G_+(n) + G_-(n)\Big] = kG_+(n+1) - kG_+(n) + kG_-(n-1) - kG_-(n) - k\alpha\beta F_+(n+1)F_-(n) - kG_+(n+1)G_-(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1). \tag{3.5}$$

The present concise form (3.1)–(3.5) of these equations has been acquired due to the use of some lowest local conservation laws dictated by the matrix structure (2.2) of the spectral operator $M(n|z)$ and found by the technique described in our recent work [19] (see also the pioneering work on the direct generation of local conservation laws [20]).

Anyway, according to the very method of their construction, the obtained equations (3.1)–(3.5) are said to possess the zero-curvature representation (2.1) with the spectral and evolution operators $M(n|z)$ and $B(n|z)$ given by formulas (2.2) and (2.3), respectively, where the constraint (2.4) imposed onto the fitting parameters α and β, as well as expressions (2.5)–(2.10) for the constituent parts of the evolution operator $B(n|z)$, have been taken into account. This property proves to be the key indication of the integrability [16] of the system under consideration (3.1)–(3.5) in the Lax sense.

4. Reduced Systems

The question how to fix the sampling function $c(n)$ is tantamount to the problem of imposing an additional constraint onto the five prototype field variables so that only four of them be truly independent. In general, there exist a number of variants how to select one of the admissible additional constraints [12] giving rise to one or another particular parametrization of field variables.

We begin with the reduction allowing us to define the sampling function $c(n)$ through some redundant quantity $q(n|n-1)$ and to exclude both of them simultaneously from the further consideration. The approach assumes the following parametrization:

$$F_+(n) = F_+ \exp\left[+x_+(n) - y_+(n) + q(n|n-1)\right], \tag{4.1}$$

$$F_-(n) = F_- \exp\left[-x_-(n) + y_-(n) - q(n+1|n)\right], \tag{4.2}$$

$$G_+(n) = 1 - \exp\left[+x_-(n) + y_-(n) - y_+(n)\right]\cosh[x_+(n)], \tag{4.3}$$

$$T(n) = 1 - \exp\left[-y_+(n) + y_-(n)\right]\cosh[x_+(n) + x_-(n)], \tag{4.4}$$

$$G_-(n) = 1 - \exp\left[-x_+(n) - y_+(n) + y_-(n)\right]\cosh[x_-(n)], \tag{4.5}$$

where $\dot{F}_+ = 0 = \dot{F}_-$.

The equations of motion for the field variables $x_+(n)$, $y_+(n)$ and $x_-(n)$, $y_-(n)$ read as follows:

$$\dot{x}_+(n) = kG_+(n+1) - kG_+(n), \tag{4.6}$$

$$\dot{y}_+(n) = kT(n) + k\alpha\beta F_+(n+1)F_-(n) + $$
$$+ kG_+(n+1)G_-(n) - k\alpha\beta F_+F_-, \tag{4.7}$$

$$\dot{x}_-(n) = kG_-(n-1) - kG_-(n), \tag{4.8}$$

$$\dot{y}_-(n) = kT(n) + k\alpha\beta F_+(n)F_-(n-1) + $$
$$+ kG_+(n)G_-(n-1) - k\alpha\beta F_+F_-. \tag{4.9}$$

These equations are seen to be essentially self-consistent, i.e., they contain neither the sampling function $c(n)$, nor the redundant variable $q(n|n-1)$. As for the variable $q(n|n-1)$, it serves mainly for the definition of a sampling function $c(n)$ by means of the equation

$$\dot{q}(n|n-1) = c(n) + k\alpha\beta F_+(n)F_-(n-1) + $$
$$+ kG_+(n)G_-(n-1) - k\alpha\beta F_+F_-. \tag{4.10}$$

Nevertheless, namely the proper choice of this definition (4.10) ensures the proper frame of reference for the true field variables $x_+(n)$, $y_+(n)$ and $x_-(n)$, $y_-(n)$ due to the presence of the last term on the right-hand sides of Eqs. (4.7) and (4.9) for $\dot{y}_+(n)$ and $\dot{y}_-(n)$.

The structure of Eqs. (4.7), (4.9), and (4.10) prompts us to adopt $F_+F_- = 1$ without any loss of generality.

We call the obtained system (4.6)–(4.9) supplemented by the parametrization formulas (4.1)–(4.5) as the reduced integrable system in symmetric parametrization. While having been convenient for the mathematical consideration, this reduction appears to be less suitable for the physical interpretation.

Fortunately, there exists an alternative reduction closely related to the ladder-like nonlinear electric transmission lines [21–24] and exhibiting the same low-amplitude behavior as the symmetric one (4.6)–(4.9). The reduction is based on the constraint

$$\frac{d}{d\tau} \ln \frac{F_+(n)F_-(n)}{[1-T(n)]^2 - [G_+(n) - G_-(n)]^2} = 0 \tag{4.11}$$

that fixes the sampling function $c(n)$ by the expression

$$c(n) = c + kG_+(n) + kG_-(n-1), \tag{4.12}$$

with c being an arbitrary function of time τ. Then, introducing the new true field variables $g_+(n)$, $g_-(n)$ and $t(n)$, $w(n)$ by means of the substitutions

$$G_+(n) = g_+(n) + [1 - g_+(n)]t(n), \tag{4.13}$$

$$G_-(n) = g_-(n) + [1 - g_-(n)]t(n) \tag{4.14}$$

and

$$T(n) = [1 - g_+(n)g_-(n)]t(n) + g_+(n)g_-(n), \tag{4.15}$$

$$F_+(n) = F_+ \exp[+w(n)] \times $$
$$\times [1 - g_+(n)][1 - t(n)][1 + g_-(n)], \tag{4.16}$$

$$F_-(n) = F_- \exp[-w(n)] \times $$
$$\times [1 + g_+(n)][1 - t(n)][1 - g_-(n)] \tag{4.17}$$

(with $\dot{F}_+ = 0 = \dot{F}_-$) and specifying the general integrable system (3.1)–(3.5) by formula (4.12) for the sampling function $c(n)$, we have

$$\frac{\dot{g}_+(n)}{1 - g_+^2(n)} = kG_-(n) - kG_-(n-1), \tag{4.18}$$

$$\frac{\dot{g}_-(n)}{1 - g_-^2(n)} = kG_+(n+1) - kG_+(n) \tag{4.19}$$

and

$$\frac{\dot{t}(n)}{1 - t(n)} = $$
$$= k\alpha\beta F_+(n+1)F_-(n) - k\alpha\beta F_+(n)F_-(n-1) + $$
$$+ kG_+(n+1)G_-(n) - kG_+(n)G_-(n-1) - $$
$$- kG_+(n+1)g_-(n) + kg_+(n)G_-(n-1) + $$
$$+ kG_+(n)g_-(n) - kg_+(n)G_-(n), \tag{4.20}$$

$$\dot{w}(n) = kG_+(n) + kG_-(n) - kT(n). \tag{4.21}$$

Here, we have assumed without any loss of generality that $c = 0$. System (4.18)–(4.21) as a whole describes two coupled subsystems so that the first two equations (4.18) and (4.19) correspond to the subsystem of a nonlinear self-dual electric network, while the last two equations (4.20) and (4.21) are related to some nonlinear vibrational subsystem.

In order to explain the above statement, let us consider the simplest case where the coupling parameter $\alpha\beta$ is equal to zero: $\alpha\beta = 0$. Then it is possible to satisfy the third equation (4.20) by putting $t(n) = 0$. As a consequence, the first two equations (4.18) and (4.19) become essentially self-consistent,

$$\frac{\dot{g}_+(n)}{1 - g_+^2(n)} = kg_-(n) - kg_-(n-1), \tag{4.22}$$

$$\frac{\dot{g}_-(n)}{1 - g_-^2(n)} = kg_+(n+1) - kg_+(n), \tag{4.23}$$

while the fourth equation (4.21) becomes redundant.

Provided the quantity $g_+(n)$ is identified with the dimensionless current $I(n)$ through the n-th inductor and the quantity $g_-(n)$ with the dimensionless voltage $V(n)$ on the n-th capacitor, the truncated system (4.22) and (4.23) can be treated as a nonlinear ladder-like electric network system, whose electric scheme is presented in a number of classical articles [21–24]. However, the functional characteristics $L(I(n))$ and $C(V(n))$ of inductors and capacitors in our system (4.22) and (4.23) are distinct from those adopted in either of the mentioned publications [21–24]. Precisely, the functional dependences corresponding to our truncated system (4.22) and (4.23) are as follows:

$$L(I(n)) = \frac{\mathrm{artanh} I(n)}{I(n)} \tag{4.24}$$

and

$$C(V(n)) = \frac{\mathrm{artanh} V(n)}{V(n)}\,. \tag{4.25}$$

5. Low-Amplitude Quartic Dispersion Equation and General Principles of Its Analysis

Assuming the coupling parameters k and $\alpha\beta$ to be the real ones, let us obtain the dispersion equation for low-amplitude excitations in the symmetrically reduced semidiscrete nonlinear system (4.6)–(4.9). In so doing, we linearize the system of our interest (4.6)–(4.9) by expanding its left-hand-side terms near the values $x_+(n) = 0$, $y_+(n) = 0$ and $x_-(n) = 0$, $y_-(n) = 0$ and use the standard plane-wave ansätze

$$x_+(n) = x^+ \exp(i\varkappa n - i\omega\tau), \tag{5.1}$$

$$y_+(n) = y^+ \exp(i\varkappa n - i\omega\tau), \tag{5.2}$$

$$x_-(n) = x^- \exp(i\varkappa n - i\omega\tau), \tag{5.3}$$

$$y_-(n) = y^- \exp(i\varkappa n - i\omega\tau). \tag{5.4}$$

Then the spectrum of the linearized system

$$\dot{x}_+(n)/k \approx y_+(n+1) - y_-(n+1) - x_-(n+1) - - y_+(n) + y_-(n) + x_-(n), \tag{5.5}$$

$$\dot{y}_+(n)/k \approx y_+(n) - y_-(n) + \alpha\beta\big[x_+(n+1) - - y_+(n+1) - x_-(n) + y_-(n)\big], \tag{5.6}$$

$$\dot{x}_-(n)/k \approx y_+(n-1) - y_-(n-1) + x_+(n-1) - - y_+(n) + y_-(n) - x_+(n), \tag{5.7}$$

$$\dot{y}_-(n)/k \approx y_+(n) - y_-(n) + \alpha\beta\big[x_+(n) - - y_+(n) - x_-(n-1) + y_-(n-1)\big] \tag{5.8}$$

will be determined by the quartic dispersion equation

$$\Omega^4 - 2\alpha\beta\sin(\varkappa)\Omega^3 - 2[1-\cos(\varkappa)]\Omega^2 + + 2\alpha\beta[1 - 2\cos(\varkappa)][1-\cos(\varkappa)]\Omega^2 + + 8\alpha\beta\sin(\varkappa)[1-\cos(\varkappa)]\Omega - - 4\alpha\beta[1-\cos(\varkappa)]^2 = 0, \tag{5.9}$$

where we have introduced the normalized cyclic frequency $\Omega = \omega/k$ and restricted the coupling parameter k to be time-independent. It is remarkable that the same dispersion equation (5.9) can be obtained also in the low-amplitude limit of an alternative physically motivated reduction (4.18)–(4.21).

To examine the roots of Eq. (5.9), it is appropriate to make the substitution

$$\Omega = 2\lambda\sin(\varkappa/2). \tag{5.10}$$

As a consequence, we come to the quartic auxiliary dispersion equation

$$\lambda^4 + a(\varkappa|\alpha\beta)\lambda^3 + b(\varkappa|\alpha\beta)\lambda^2 + c(\varkappa|\alpha\beta)\lambda + d(\varkappa|\alpha\beta) = 0, \tag{5.11}$$

where

$$a(\varkappa|\alpha\beta) = -2\alpha\beta\cos(\varkappa/2), \tag{5.12}$$

 ISSN 2071-0186. Ukr. J. Phys. 2014. Vol. 59, No. 6

$$b(\varkappa|\alpha\beta) = -1 + 3\alpha\beta - 4\alpha\beta\cos^2(\varkappa/2), \tag{5.13}$$

$$c(\varkappa|\alpha\beta) = +4\alpha\beta\cos(\varkappa/2), \tag{5.14}$$

$$d(\varkappa|\alpha\beta) = -\alpha\beta. \tag{5.15}$$

Its discriminant [25, 26] $D(\varkappa|\alpha\beta)$ is given by the formula [25, 27, 28]

$$D(\varkappa|\alpha\beta) = \\ = a^2(\varkappa|\alpha\beta)b^2(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta) - 4a^3(\varkappa|\alpha\beta)c^3(\varkappa|\alpha\beta) - \\ - 4b^3(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta) + 18a(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta)c^3(\varkappa|\alpha\beta) - \\ - 27c^4(\varkappa|\alpha\beta) + 256d^3(\varkappa|\alpha\beta) - \\ - 4a^2(\varkappa|\alpha\beta)b^3(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) + \\ + 18a^3(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) + \\ + 16b^4(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) - \\ - 80a(\varkappa|\alpha\beta)b^2(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) - \\ - 6a^2(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) + \\ + 144b(\varkappa|\alpha\beta)c^2(\varkappa|\alpha\beta)d(\varkappa|\alpha\beta) - 27a^4(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta) + \\ + 144a^2(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta) - \\ - 128b^2(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta) - \\ - 192a(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta)d^2(\varkappa|\alpha\beta). \tag{5.16}$$

In the regions of a negative discriminant $D(\varkappa|\alpha\beta) < 0$, the theory of quartic equations [25–27] predicts two real roots and two complex roots. However, in the regions of a positive discriminant $D(\varkappa|\alpha\beta) > 0$, the situation turns out to be ambiguous until we invoke two seminvariants $H(\varkappa|\alpha\beta)$ and $Q(\varkappa|\alpha\beta)$ given by the formulas [27, 28]

$$H(\varkappa|\alpha\beta) = 8b(\varkappa|\alpha\beta) - 3a^2(\varkappa|\alpha\beta), \tag{5.17}$$

$$Q(\varkappa|\alpha\beta) = 3a^4(\varkappa|\alpha\beta) - 16a^2(\varkappa|\alpha\beta)b(\varkappa|\alpha\beta) + \\ + 16a(\varkappa|\alpha\beta)c(\varkappa|\alpha\beta) + 16b^2(\varkappa|\alpha\beta) - \\ - 64d(\varkappa|\alpha\beta) \tag{5.18}$$

and determine their signs. Precisely at $D(\varkappa|\alpha\beta) > 0$, the general theory [25–28] predicts four real roots provided $H(\varkappa|\alpha\beta) < 0$ and $Q(\varkappa|\alpha\beta) > 0$ or four complex roots provided either $H(\varkappa|\alpha\beta) > 0$ or $Q(\varkappa|\alpha\beta) < 0$.

6. Typical Features of the Low-Amplitude Dispersion Law as a Function of the Adjustable Coupling Parameter $\alpha\beta$

Inasmuch as the quantities $D(\varkappa|\alpha\beta)$, $H(\varkappa|\alpha\beta)$ and $Q(\varkappa|\alpha\beta)$ depend only on two parameters $\varkappa$ and $\alpha\beta$, it is convenient to use the plane of these parameters in order to visualize the regions, where the signs of $D(\varkappa|\alpha\beta)$, $H(\varkappa|\alpha\beta)$ and $Q(\varkappa|\alpha\beta)$ remain fixed. Figure 1 shows such regions found by the computer simulation. Each region is marked by three vertically arranged signs (column of three signs) so that the upper, middle, and lower signs are related, respectively, to $D(\varkappa|\alpha\beta)$, $H(\varkappa|\alpha\beta)$, and $Q(\varkappa|\alpha\beta)$. N.B. The caudal-fin like region (although being unlabeled due to the lack of space) is understood as marked by col$(+-+)$ signature.

At $\varkappa = 0$ and $\varkappa = \pm\pi$, the results presented in Fig. 1 allow the direct analytical interpretation based on the simplified expressions

$$D(0|\alpha\beta) = 16\alpha\beta(2\alpha\beta - 1)^2 \times \\ \times [17(\alpha\beta)^3 + 33(\alpha\beta)^2 - 12\alpha\beta - 1], \tag{6.1}$$

$$H(0|\alpha\beta) = -4[(\alpha\beta + 1)^2 + 2(\alpha\beta)^2 + 1], \tag{6.2}$$

$$Q(0|\alpha\beta) = 16[3(\alpha\beta)^4 + 4(\alpha\beta)^3 - 3(\alpha\beta)^2 + 6\alpha\beta + 1] \tag{6.3}$$

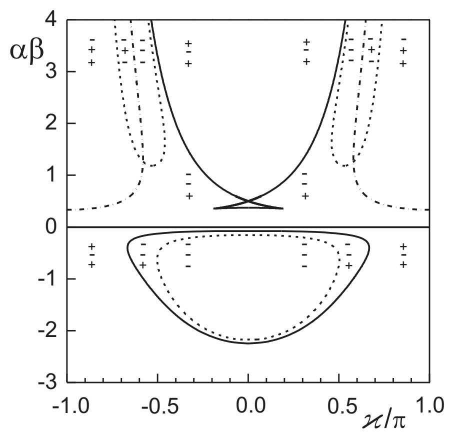

Fig. 1. Subdivision into the regions with fixed signs of $D(\varkappa|\alpha\beta)$ (upper sign), $H(\varkappa|\alpha\beta)$ (middle sign), $Q(\varkappa|\alpha\beta)$ (lower sign) in the plane of the wave vector $\varkappa$ and the adjustable coupling parameter $\alpha\beta$. The curves $D(\varkappa|\alpha\beta) = 0$ are marked by the solid lines. The curves $H(\varkappa|\alpha\beta) = 0$ are marked by the dot-dashed lines. The curves $Q(\varkappa|\alpha\beta) = 0$ are marked by the dotted lines

and

$$D(\pm\pi|\alpha\beta) = -16\alpha\beta[(\alpha\beta-1)^2 + 8(\alpha\beta)^2], \tag{6.4}$$

$$H(\pm\pi|\alpha\beta) = 8(3\alpha\beta - 1), \tag{6.5}$$

$$Q(\pm\pi|\alpha\beta) = 16[(\alpha\beta-1)^2 + 8(\alpha\beta)^2] \tag{6.6}$$

following from the general ones (5.12)–(5.18).

Thus, at $\varkappa = 0$, formula (6.1) for $D(0|\alpha\beta)$ ensures that all six roots of the equation $D(0|\alpha\beta) = 0$ must be purely real. To wit, we have

$$(\alpha\beta)_1 \simeq -2.2441, \tag{6.7}$$

$$(\alpha\beta)_2 \simeq -0.0703, \tag{6.8}$$

$$(\alpha\beta)_3 = 0, \tag{6.9}$$

$$(\alpha\beta)_4 \simeq 0.3731, \tag{6.10}$$

$$(\alpha\beta)_5 = 0.5, \tag{6.11}$$

$$(\alpha\beta)_6 = 0.5, \tag{6.12}$$

where the roots $(\alpha\beta)_3$, $(\alpha\beta)_5$, and $(\alpha\beta)_6$ are self-evident, while the roots $(\alpha\beta)_1$, $(\alpha\beta)_2$, and $(\alpha\beta)_4$ are obtainable from the cubic equation, whose discriminant [25, 27] is proved to be positive. As for the quantity $H(0|\alpha\beta)$ (formula (6.2)), it is seen to be essentially negative. At last, the quartic equation $Q(0|\alpha\beta) = 0$ (see formula (6.3) for $Q(0|\alpha\beta)$) must possess two real negative roots, since its discriminant is negative and $Q(0|\alpha\beta \geq 0) > 0$, while the parameter $\alpha\beta$ must be purely real by definition.

The consideration of formulas (6.4)–(6.6) for $D(\pm\pi|\alpha\beta)$, $H(\pm\pi|\alpha\beta)$, and $Q(\pm\pi|\alpha\beta)$ related to $\varkappa = \pm\pi$ yields $D(\pm\pi|\alpha\beta > 0) < 0$, $D(\pm\pi|\alpha\beta < 0) > 0$, and $H(\pm\pi|\alpha\beta > 1/3) > 0$, $H(\pm\pi|\alpha\beta < 1/3) < 0$, whereas $Q(\pm\pi|\alpha\beta) > 0$ for all real $\alpha\beta$.

Examining the analytical data contained in three previous paragraphs, we clearly observe their one-to-one reproductions on the lines $\varkappa = 0$ and $\varkappa = \pm\pi$ of Fig. 1.

According to the commonly accepted graphical treatment of a dispersion law [29–31], we shall be interested in the real-valued solutions $\lambda_j^*(\varkappa|\alpha\beta) = \lambda_j(\varkappa|\alpha\beta)$ of the auxiliary dispersion equation (5.11) at the real-valued wave vector $\varkappa$ spanning the first Brillouin zone $-\pi \leq \varkappa \leq +\pi$. Thus, juxtaposing the signatures of all regions pictured in Fig. 1 with the early listed criteria on the character of roots, we can readily conclude that the regions marked by col$(+-+)$ signature should produce the four-branch auxiliary dispersion law, while the auxiliary dispersion law in the other regions should be the two-branch one.

The same statement concerns also the actual dispersion law, i.e., the dispersion law given in terms of cyclic eigenfrequencies $\Omega_j(\varkappa|\alpha\beta) = 2\lambda_j(\varkappa|\alpha\beta)\times \sin(\varkappa/2)$, except for the merging point $\varkappa = 0$, where $\Omega_j(\varkappa = 0|\alpha\beta) \equiv 0$. Here, the integer j denotes the ordinal number of the eigenmode of low-amplitude excitations and serves as the branch number in their dispersion law.

Figure 2 demonstrates the typical metamorphoses in the actual low-amplitude dispersion law as the adjustable coupling parameter $\alpha\beta$ grows from the values smaller than $(\alpha\beta)_1$ to the values larger than $(\alpha\beta)_5 = (\alpha\beta)_6$. As we have expected, the most crucial qualitative rearrangements in the dispersion law are seen to happen when the value of coupling parameter $\alpha\beta$ passes through any root $(\alpha\beta)_k$ of the equation $D(0|\alpha\beta) = 0$ or through the value $(\alpha\beta)_* \simeq 0.3557$ being the ordinate for each of two symmetric casp points on the line $D(\varkappa|\alpha\beta) = 0$ (see the caudal-fin region in Fig. 1). In this respect, the points $(\alpha\beta)_k$ (with $k = 1, 2, 3, 4, 5, 6$) and $(\alpha\beta)_*$ can be referred to as the critical ones. Meanwhile, when the coupling parameter $\alpha\beta$ varies between any two distinct neighboring critical points, the changes in a structure of the dispersion law are proved to be solely quantitative rather than qualitative in complete agreement with the criteria evaluating the character of admissible roots of the quartic auxiliary dispersion equation (5.11).

Considering the dispersion curves on each subfigure of Fig. 2, we observe that some of them have the dead-end points with respect to the wave vector $\varkappa$. Nonetheless, each dead-end point is seen to be shared by two distinct branches. As a consequence, the combination of such joint branches can be treated as the unique loop-like branch or the unique folding branch. Here, we would like to point out on a certain similarity between the low-amplitude oscillations in our semidiscrete system (4.6)–(4.9) and the beam-plasma oscillations in hydrodynamic plasma, where the loop-like structure of a dispersion law is known to be rather typical [30, 31]. Regarding the linearized version (5.5)–(5.8) of our integrable system (4.6)–

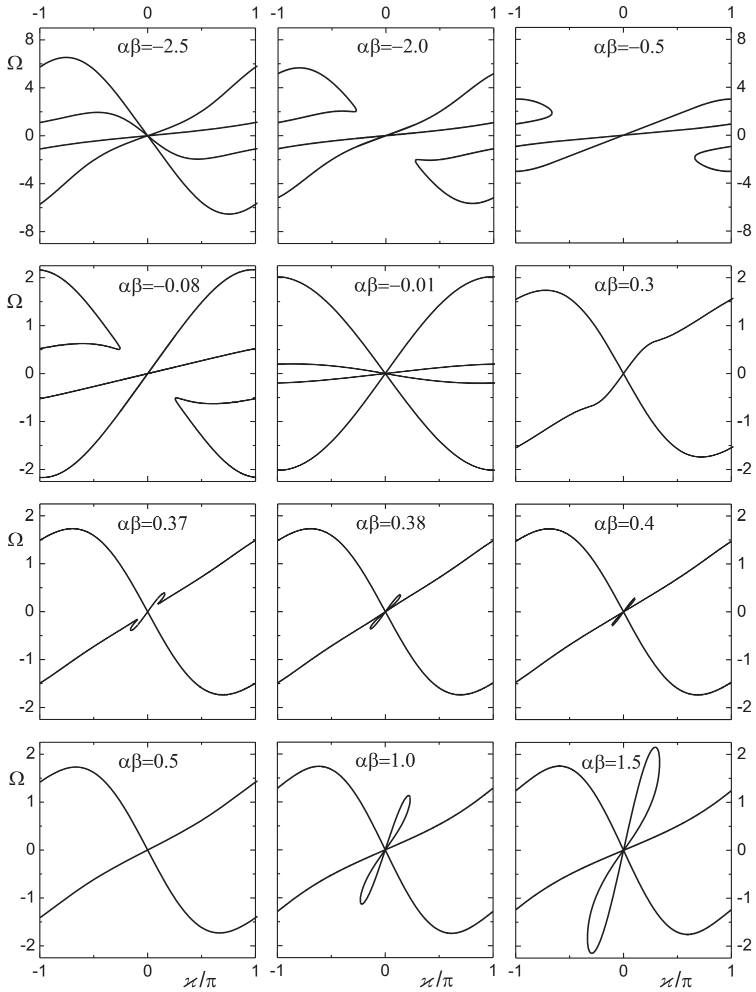

Fig. 2. Real-valued normal cyclic frequencies as a function of the wave vector at twelve distinct values of adjustable coupling parameter

(4.9), Fig. 1 prompts that the loop-like structure of a dispersion law must inevitably emerge, once the coupling parameter $\alpha\beta$ enters into one of the following two intervals:

$$(\alpha\beta)_4 < \alpha\beta < (\alpha\beta)_5 \equiv (\alpha\beta)_6 \qquad (6.13)$$

or

$$(\alpha\beta)_5 \equiv (\alpha\beta)_6 < \alpha\beta < \infty, \qquad (6.14)$$

where the critical points $(\alpha\beta)_k$ are given by formulas (6.7)–(6.12). As for the folding branch structure of a dispersion law, it must be prescribed to the interval

$$(\alpha\beta)_* < \alpha\beta < (\alpha\beta)_4, \qquad (6.15)$$

where $(\alpha\beta)_* \simeq 0.3557$ as we already know. The peculiarities concerning the loop-like and folding-like dispersion curves are clearly seen on the respective subfigures of Fig. 2.

From the standpoint of stability analysis, any interval of wave vectors supporting four real-valued branches of the quartic dispersion relation (5.9) should be treated as an interval of stability, while any interval of wave vectors supporting two real-valued branches of the quartic dispersion relation (5.9) should be considered as an interval of instability. Here, due to the spatial discreteness of our linearized system (5.5)–(5.8), it seems impossible to apply the Sturrock rules [29, 30] and to qualify each particular instability either as convective or absolute one. For us, however, the mere fact of instability is more important than the strict identification of instability type.

Fortunately, the predisposition to an instability is not a dominant property of the considered linearized system (5.5)–(5.8), since it can be eliminated by the proper choice of the adjustable coupling parameter $\alpha\beta$. Precisely at

$$-\infty < \alpha\beta < (\alpha\beta)_1 \tag{6.16}$$

or at

$$(\alpha\beta)_2 < \alpha\beta < 0, \tag{6.17}$$

all four branches of low-amplitude excitations are stable for all wave vectors $-\pi \leq \varkappa \leq +\pi$, thus ensuring the good background for the stable solutions of any semidiscrete nonlinear system of interest (4.6)–(4.9) or (4.18)–(4.21).

7. Conclusion

Basing on the semidiscrete zero-curvature equation, we have found the mutually consistent spectral and evolution operators of a new type and have obtained the early unknown integrable nonlinear evolution system on a quasi-one-dimensional lattice.

When choosing the fixation of a sampling function, one can come to distinct reductions of the basic semidiscrete system and, as a consequence, to the distinct parametrizations of prototype field amplitudes. We have considered two possible reduced semidiscrete integrable systems, one of which provides us with its clear physical interpretation as a subsystem of the ladder-like nonlinear transmission line coupled with the subsystem of nonlinear lattice vibrations.

Each integrable nonlinear system (either reduced or basic one) is characterized by two parameters responsible for the interfield couplings of principally distinct origins. The variation of coupling parameters should lead to several regimes in the behavior of the system with nontrivial dynamics. This statement finds its natural confirmation already on the stage of low-amplitude plane-wave excitations, whose dispersion law turns out to be essentially dependent on the magnitude and the sign of the adjustable coupling parameter provided other coupling parameter is fixed. Thus, in some windows of the adjustable coupling parameter, we observe the clear resemblance between the obtained loop-like dispersion curves and those typical of the beam-plasma oscillations in hydrodynamic plasma [30, 31]. There are also windows of the adjustable coupling parameter, where all branches of low-amplitude oscillations remain stable within the whole Brillouin zone.

In contrast, in a majority of already known integrable systems [1–4, 15, 22, 24], the spectrum of low-amplitude excitations remains fixed, inasmuch as the governing parameter responsible for its rearrangements is simply absent.

This work has been supported by the National Academy of Sciences of Ukraine within the Program No. 0112U000056.

1. S.V. Manakov, Sov. Phys. – JETP **40**, 269 (1975).
2. H. Flaschka, Progr. Theor. Phys. **51**, 703 (1974).
3. M.J. Ablowitz and J.F. Ladik, J. Math. Phys. **16**, 598 (1975).
4. M.J. Ablowitz and J.F. Ladik, J. Math. Phys. **17**, 1011 (1976).
5. D.N. Christodoulides and R.I. Joseph, Opt. Lett. **13**, 794 (1988).
6. C. Waschke, H.G. Roskos, R. Schwedler, K. Leo, H. Kurz, and K. Köhler, Phys. Rev. Lett. **70**, 3319 (1993).
7. V.G. Lyssenko, G. Valušis, F. Löser, T. Hasche, K. Leo, M.M. Dignam, and K. Köhler, Phys. Rev. Lett. **79**, 301 (1997).
8. P. Marquie, J.M. Bilbault, and M. Remoissenet, Phys. Rev. E **49**, 828 (1994).
9. A.S. Davydov, *Biology and Quantum Mechanics* (Pergamon Press, Oxford–New York, 1981).
10. J.W. Mintmire, B.I. Dunlap, and C.T. White, Phys. Rev. Lett. **68**, 631 (1992).
11. O.O. Vakhnenko, J. Nonlin. Math. Phys. **18**, 401 (2011).
12. O.O. Vakhnenko, J. Nonlin. Math. Phys. **18**, 415 (2011).
13. P.J. Caudrey, Physica D **6**, 51 (1982).
14. P.J. Caudrey, in *Wave Phenomena: Modern Theory and Applications*, edited by C. Rogers and T.B. Moodie (Elsevier, Amsterdam, 1984), p. 221.
15. O.O. Vakhnenko, J. Phys. A **36**, 5405 (2003).
16. L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin, 1987).
17. T. Tsuchida, J. Phys. A **35**, 7827 (2002).
18. O.O. Vakhnenko, J. Phys. A **39**, 11013 (2006).
19. O.O. Vakhnenko, J. Nonlin. Math. Phys. **20**, 606 (2013).
20. M. Wadati, H. Sanuki, and K. Konno, Progr. Theor. Phys. **53**, 419 (1975).
21. R. Hirota and K. Suzuki, Proc. of IEEE **61**, 1483 (1973).
22. R. Hirota, J. Phys. Soc. Japan **35**, 289 (1973).
23. K. Daikoku, Y. Mizushima, and T. Tamama, Jap. J. Appl. Phys. **14**, 367 (1975).
24. R. Hirota and J. Satsuma, J. Phys. Soc. Japan **40**, 891 (1976).
25. L.E. Dickson, *Elementary Theory of Equations* (Wiley, New York, 1914).
26. E.L. Rees, Amer. Math. Monthly **29**, 51 (1922).
27. J.E. Cremona, LMS J. Comput. Math. **2**, 62 (1999).

28. R.W.D. Nickalls, Math. Gazette **93**, 66 (2009).
29. P.A. Sturrock, Phys. Rev. **112**, 1488 (1958).
30. A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Sitenko, and K.N. Stepanov, *Plasma Electrodynamics. Vol. 1. Linear Theory* (Pergamon Press, Oxford–New York, 1975).
31. L.V. Postnikov, V.I. Korolev, T.M. Tarantovich, V.A. Mel'nikova, and S.Ya. Vyshkind, Collection of Problems of the Theory of Oscillations (Nauka, Moscow, 1978) (in Russian).

Received 18.06.12

О.О. Вахненко, В.О. Вахненко

ЛІНІЙНИЙ АНАЛІЗ РОЗШИРЕНОЇ ІНТЕҐРОВНОЇ МОДЕЛІ НЕЛІНІЙНОЇ ДРАБИНЧАСТОЇ МЕРЕЖІ

Р е з ю м е

Представлено нове нетривіальне інтеґровне розширення нелінійної моделі драбинчастої електричної мережі, яке характеризується двома параметрами зв'язку. Спираючись на декілька найнижчих локальних законів збереження, подано стислу форму загальної напівдискретної інтеґровної системи та знайдено дві версії її самодостатньої редукції в термінах чотирьох істинних польових змінних. Проведено всебічний аналіз дисперсійного рівняння низькоамплітудних збуджень системи. Сформульовано критерії, що вирізняють двогілкову та чотиригілкову реалізації закону дисперсії. Встановлено критичні значення регульовного параметра зв'язку та проілюстровано низку якісно відмінних реалізацій закону дисперсії. Вказано, що петлеподібна будова низькоамплітудного закону дисперсії редукованої системи, притаманна певним вікнам регульовного параметра зв'язку, відтворює петлеподібну структуру типового закону дисперсії струменево-плазмових коливань в гідродинамічній плазмі. Очікується, що розмаїття низькоамплітудного спектра запропонованої драбинчастої мережі як функції регульовного параметра зв'язку здатне привести до ще більшого розмаїття динамічної поведінки системи в суттєво нелінійному режимі.